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Visualization of the Gram–Schmidt Orthogonalization process with Geogebra

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RABSTRACT

This study presents the development of an activity carried out in the classroom within the Analytical Geometry course of the bachelor's degree in Exact and Technological Sciences at the Federal University of Recôncavo da Bahia, Cruz das Almas campus. In this activity, the GeoGebra software was used as a tool to investigate students' learning perceptions regarding the concepts of bases and the Gram–Schmidt orthogonalization process. The objective of this article is to present the results of this investigation, analyzing how the use of digital technology can contribute to the understanding of these concepts. The results indicate that, although some students experienced difficulties with both the use of the software and the theoretical content of the course, the activity was able to enhance the mathematical understanding of most students. Thus, the activity proved to be a viable alternative for the integration of Digital Technologies in mathematics teaching, particularly in the teaching of Analytical Geometry.

KEYWORDS: Analytical geometry. Digital technology. Mathematics education.

1. INTRODUCTION

The Analytical Geometry subject is included in several courses in the exact sciences area. One of the contents addressed in that subject is the Gram-Schmidt orthogonalization process, which is an algorithm used to obtain an orthogonal base (or orthonormal) of vectors from any base. This method allows the transformation of a set of three linearly independent vectors into an orthogonal (or orthonormal) set of vectors forming a V^3 base, where V^3 is the space of vectors (Boulos; Camargo, 2005). Students usually have objections to this content since the process formulas involve several mathematical calculations and, therefore, few manage to reach a geometrical understanding of the orthogonalization process.

Research linked to Mathematics Education has emphasized the importance of visualization and visual reasoning in mathematics teaching and learning, mainly regarding geometry (Corradi; Franco, 2020). Some authors such as Gutiérrez (1996) and Duval (2015), described distinct conceptions of visualization; however, both indicate it as a process to be incentivized and that goes beyond the simple act of seeing or perception in itself, and involves the ability to interpret, reflect upon and use visual information significantly, thus favoring learning.

Mathematical visualization is defined as the "process of image formation (mental, with pencil and paper or the use of technologies) using such images in an efficient manner to discover and understand mathematics" (Zimmermann; Cunningham, 1991, p. 3). Therefore, visualization is not seen as an end in itself, but rather as a resource that optimizes the understanding of mathematical concepts (Flores; Wagner; Buratto, 2012).

> Many authors highlight the contributions that the use of technology can bring to the educational process, facilitating the visualization of what is happening, favoring the construction of knowledge by the student, enabling teacher-student interaction, and assisting the teacher's work. When used creatively, new technologies can boost student participation in classes, as young people are naturally interested in technology; thus, it is up to the teacher to channel this interest to promote the classroom as a space for active learning and reflection. The student is fascinated by the interaction that arises and by the possibility of participating, suggesting ideas and asking questions on the computer itself without having to go to the teacher. They have the possibility of testing various data variations, obtaining answers to their questions and doubts that arise during the activities (Guedes, 2015, p. 366).

According to Maia, Gondim and Vasconcelos (2023), transforming the classroom into an interactive environment with the use of the GeoGebra software, more specifically in mathematics teaching, is a learning strategy that allows students to interact and form concepts, in addition to contributing to the overcome of difficulties in geometry learning observed through evaluations.

On the other hand, a study conducted by Yohannes and Chen (2021) surveyed articles in the *Web of Science* database, from 2010 to 2020, which were relevant for the GeoGebra software integration in mathematics education. The research participants in those articles were elementary, high school and higher education students. Few studies were seen to investigate cognitive load, learning anxiety and students' involvement. Although GeoGebra is an open code, that study also revealed that only a limited number of countries is integrating GeoGebra in mathematics education.

According to Miskulin (1999), when discussing the possibility of using software in education, one of the concerns is how such technological resources can be used in the best way possible to contribute to the teaching/learning process. In other words, teachers must always reflect upon the possibilities of the software to develop mental processes.

Furthermore, the use of Digital Technologies (DT) in Brazilian universities faces practical obstacles, mainly regarding the lack of material resources. However, the possibilities created by technological and computational resources such as GeoGebra software use, should not be neglected. According to reports by Giraldo, Caetano and Mattos (2012), teachers should know such possibilities so that they can develop counter-arguments to the existing difficulties. Moreover, as pointed out by Borba and Villarreal (2005), DTs work as tools that promote exploration and experimentation, thus favoring students' autonomy, curiosity and a dynamic approach to knowledge production.

Within such perspective, the construct "humans with media" broadens such understanding by considering DTs as co-participants in the production of mathematics knowledge, not only as ancillary tools. According to Borba (2012), Borba, Silva and Gadanidis (2014), Borba *et al.* (2016) and Borba, Souto and Canedo Jr. (2022), this construct development follows the transformations occurred in digital technologies, enabling the identification of different phases of their insertion in mathematics education. These transformations have been used to understand how these technologies reconfigure the classroom and the learning processes.

Therefore, the use of technological resources in education is one of UNESCO concerns (United Nations Educational, Scientific and Cultural Organization) regarding school literacy. The organization supports the incentive to the acquisition of computer use basic skills for all and the broadened implementation of the use of Information and Communication Technologies (ICT) aimed at sustainable development and peace with a greater focus on teacher education to make them aware of the importance of media and information literacy in the education process. Therefore, UNESCO promotes the formation of literate societies in media and information such as libraries, archives, museums and the internet, regardless of the technologies used (UNESCO, 2023).

In such context and aiming at favoring students' learning in the Analytical Geometry subject, this article proposes the GeoGebra software use when teaching the Gram-Schmidt orthogonalization process. GeoGebra is a dynamic mathematics program for all levels of education, which joins geometry, algebra, spreadsheets, graphs, statistics and calculations in a single online platform with several free resources (GeoGebra, 2023).

The orthogonalization process is based on the use of vector algebra — sum of vectors and multiplication of a number by a vector — and on the concept of orthogonal projection of vectors (Boulos; Camargo, 2005). Both the Gram-Schmidt orthogonalization process formulas and those of the orthogonal projection are analytical expressions, and the absence of geometrical understanding might prevent the full understanding of those concepts. For Duval (2003), Geometry learning favors three different forms of the cognitive process, namely, visualization, construction and reasoning — which are inter-related to provide students with the necessary proficiency in Geometry.

This article aims to present the results of an investigation conducted in a class of students enrolled in the Analytical Geometry subject in the Bachelor of Exact and Technological Sciences course (BCET) at the Federal University of Recôncavo da Bahia (UFRB). It focuses on the students' perception of learning in relation to the orthogonal bases content and the Gram-Schmidt orthogonalization process using digital technology to assist the mathematical construction and content visualization.

2. METHODOLOGICAL PROCEDURES

This article adopted a qualitative, applied approach of the participant observation type. According to Souza, Lopes, and Souza (2015), this type of methodology involves an emerging and evolving project, so it does not necessarily need to be previously described. Data collection occurs through the subject's interactions with the researcher in the natural environment, in the field. This methodology is centered on the participants' perspectives. However, in this case, the analysis process is conditioned by the subject's actions during the interaction with the researcher. It usually takes place in an education environment where the researcher is involved, and it is common for the researcher to be responsible for conducting the education activities, even if these are developed collaboratively.

Aiming at improving the teaching and learning process and promoting greater students' engagement in the Analytical Geometry lessons, the researcher-professor planned and applied activities integrating DT use in the content worked. The tools chosen for this purpose was the GeoGebra software due to its potential for visual exploration and dynamic manipulation of three-dimensional objects.

In those activities, students had their first contact with the software, favoring their familiarization with the tool. Thus, the target audience of the activity proposed in this study were students who had already taken part in previous introductory activities and who, therefore, possessed the minimum conditions to use GeoGebra.

The activity application, central point of this study, was carried out on April 4, 2024, at 4pm, at the Computer Laboratory of the Classroom Pavillion 1, for students in the BCET course of the Exact and Technological Sciences Center at UFRB, Campus Cruz das Almas, enrolled in the Analytical Geometry subject, a class of approximately 40 students. The Computer Laboratory had 28 desktop computers, a white board and a slide projector.

According to the schedule, researchers had access to the Computer Laboratory and, alongside the monitor, turned on the computers to receive the students. The activity started with the presentation of the research project and the Free and Informed Consent form (TCLE, Termo de Consentimento Livre e Esclarecido). The research project that originated this article was submitted to the Research Ethics Committee at UFRB and approved with Ethical Appreciation Presentation Certificate (CAAE, Certificado de Apresentação de Apreciação Ética) number 71406123.9.0000.0056.

Next, the professor responsible for the subject presented the activity and made himself available to solve any doubts. Students started the activity around 04:30pm, and could work until 06:30pm to finish it. The professor responsible for the course was in the laboratory alongside a collaborating professor on the research project, a student teaching assistant for the course, and 26 participant students (Figure 1). The activities started with the description of the Gram-Schmidt Orthogonalization process in GeoGebra.

Figure 1 – Students during the development of the activity.

Source: The authors (2024).

The base content and the Gram-Schmidt orthogonalization process were previously taught to the class by the professor of the subject in previous lessons, enabling students to access the material made available by the professor, namely, slides, lesson notes and list of exercises. In addition, students could use the GeoGebra 3D Calculator to accomplish the subject task. Before the development of the activity, students received an access link, via Academic Activity Management System (SIGAA, Sistema de Gestão de Atividades Acadêmicas), which directed them to the task.

Data collection was carried out using four structure questions with geometrical constructions, employing the GeoGebra software and an essay question about the students' perception of the proposed activity. Students had access to the activity using a link generated by the GeoGebra 3D Calculator, which directed them to an interactive classroom where they could work on the activity (Figure 2). When starting the activity, students found the structure presented in Chart 1.

Figure 2 – GeoGebra interface when opening the activity

GeoGebra All changes were recorded

Gram-Schmidt orthogonalization process

Introduction

In this activity, we are going to explore the Gran-Schmidt orthogonalization process using the Geogebra 3D calculator software. This robust method allows users to build a set of orthogonal (or orthonormal) vectors from a set of linearly independent vectors. The activity is proposed as follows:

. Introduction to the Geogebra 3D calculator;
. Questions about the base content in the space of vectors V^3;
. Application of the Gram-Schmidt orthogonalization process.

Introduction to the Geogebra 3D calculator:

We will use the Geogebra 3D calculator to represent points, exploring some basic Commands that will be employed during the development of the activity.

Commands

. To insert a vector **u = (2,1,2)**
. To insert a point **P = (5, 0, 1)**
. To calculate a vector norm (vector length): **Length(object)**
. To obtain the angle between two vectors: **Angle (Vector,Vector)**
. To calculate the scalar product: **ScalarProduct (Vector, Vector)**
. To insert a plane: **Plane (Point, Vector, Vector)**

Source: The authors (2024).

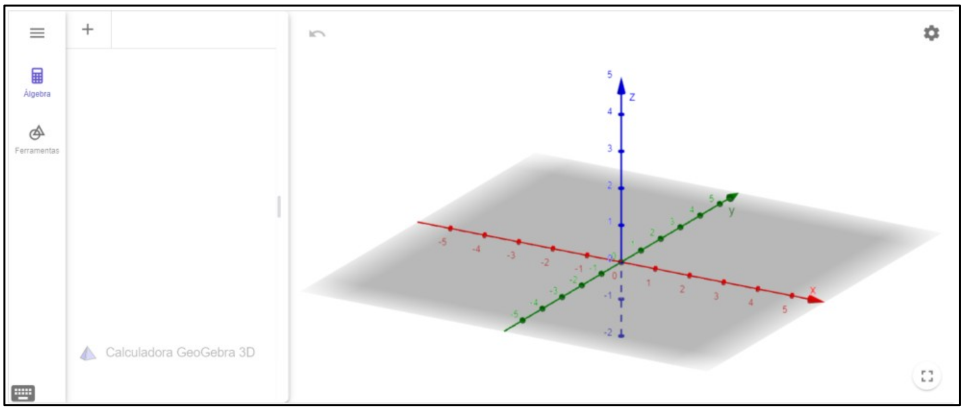
Chart 1 – Moments of Activity.

Moments	Description
Moment 1	Activity and GeoGebra 3D calculator use introduction
Moment 2	Presentation of questions about the base content in the space of Vectors V^3
Moment 3	Presentation of questions about the Gram-Schmidt orthogonalization process
Moment 4	Essay question about students' perception of the activity proposed

Source: The authors (2026).

During the introduction, basic commands to be used in the activity were presented. These included: dot, vector and plane insertion; and calculation of the vector length, angle between vectors and scalar product between vectors. In the remaining moments of the activity, questions were presented (Chart 2) when, for each question, a graphic window of the GeoGebra 3D Calculator was made available to help the resolution and geometrical visualization of the question (Figure 3).

Figure 3 – GeoGebra 3D Calculator Graphic window.



Source: The authors (2024).

Chart 2 – Instructions of the questions asked to the students.

Questions	Instruction
Question 1	Insert any non-null vectors in the GeoGebra 3D Calculator. Determine whether these vectors form a base. Justify your answer in full, using geometrical and analytical arguments.
Question 2	Insert a base for V^3 in the GeoGebra 3D Calculator, where V^3 is the vector space. Determine whether this base is orthogonal or orthonormal and justify your answer in full, using geometrical and analytical arguments.
Question 3	According to the Gram-Schmidt orthogonalization process, the base (v1,v2,v3) is orthogonal. How can you use the GeoGebra software to verify whether this base is really orthogonal? Verify it.
Question 4	According to the Gram-Schmidt orthonormalization process, the base (e1,e2,e3) is orthonormal. How can you use the GeoGebra software to verify whether this base is really orthonormal? Verify it.
Question 5	Did you find it difficult to accomplish the task? If yes, what kind of difficulty? How do you evaluate this activity contribution to your understanding of the content studied? Please, share any praise, critique or suggestion you might have

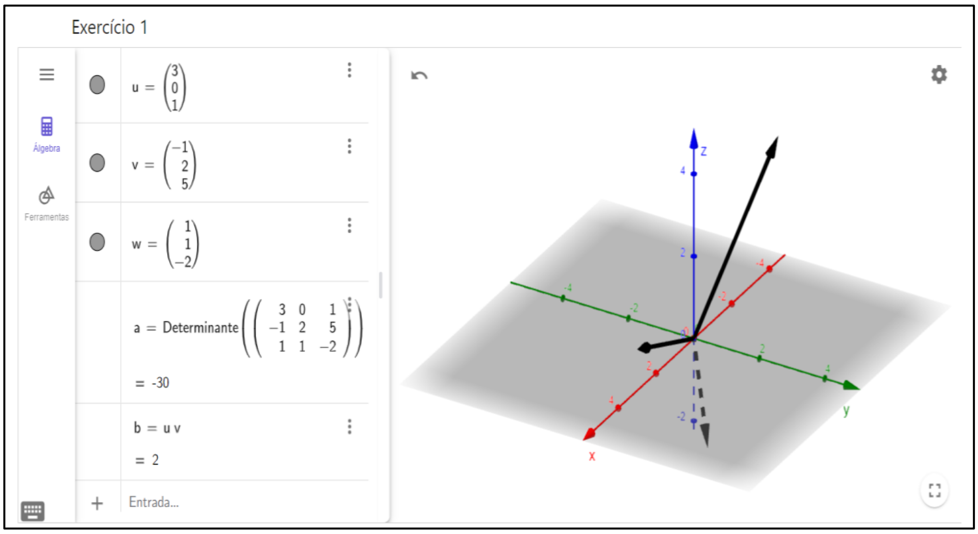
Source: The authors (2024).

The data analysis was carried out in two complementary phases. First, the answers produced by students in the GeoGebra 3D platform were extracted and organized according to each question proposed in the activity. Next, those answers were grouped by similarity, considering patterns of understanding, types of errors and strategies used by the students during the resolution process. In the second phase, qualitative analysis was carried out, focused on identifying difficulties, conceptual advances and potential contributions of the GeoGebra use in the learning of the Gram-Schmidt Orthogonalization process.

3. RESULTS AND DISCUSSION

All students who answered questions 1 and 2 about the base content in the vector space, V³, used the GeoGebra 3D Calculator widow as an ancillary tool in their answers. However, most of them only presented analytical arguments to answer Question 1. For example, Student 1 (A1) answered: "*When calculating the determinant of vectors u=(3,0,1), v=(-1,2,5) and w=(1,1,-2), the result obtained was -30, that is, ≠ 0, confirming the fact that they were linearly independent, thus forming a base.*" The same student used the window to build Figure 4:

Figure 4 – Student 1's GeoGebra 3D Calculator window for question 1.

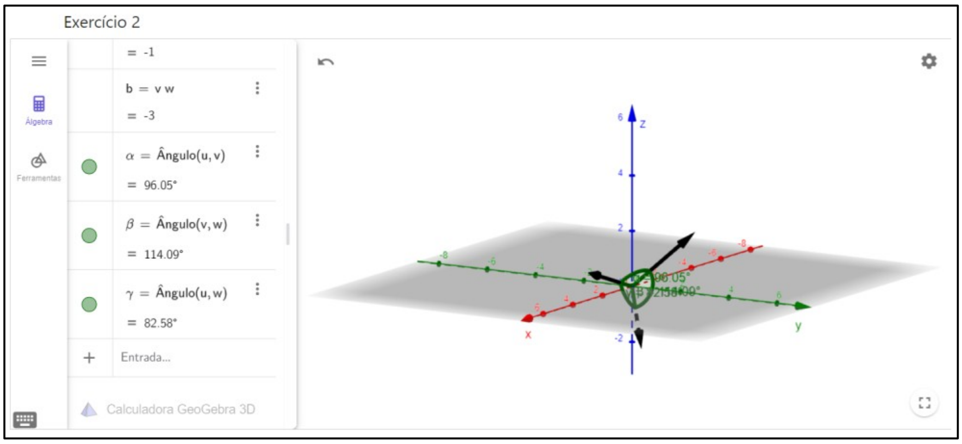


Source: the authors (2024).

We observed that the student used the determinant calculation as argument, thus revealing an analytical approach. Although Figure 4 provides a graphic representation of vectors, the student did not use the geometrical argument that, for example, the three vectors are not parallel to the same plane.

In Question 2, a possible analytical argument could be the calculation of the scalar product of vectors two by two, while a geometrical argument would be the verification of the straight angle between any two vectors. We also observed that approximately 39% of the students prioritized the geometrical argument. This is due to the fact that angle verification is easily visualized geometrically in the GeoGebra software, as answered by Student 2 (A2): "Having observed that vectors u, v, and w form a basis V³, but are not orthogonal, as they do not have a 90º angle, nor orthonormal, as none of the angles form 90º". The following representation was constructed in the GeoGebra 3D Calculator (Figure 5).

Figure 5 - Student 2's GeoGebra 3D Calculator window for question 2



Source: The authors (2024).

Also in Question 2, 19% of the students presented both geometrical and analytical arguments. One example of answer is that of Student 3 (A3): "*The base is not orthogonal, since the vectors do not form a 90° angle between them, and the scalar product is different from 0. It is not orthonormal either, because the vectors are not unitary*".

Approximately 27% of the students did not produce correct and/or satisfactory answers, while 15% of them used only analytical arguments such as Student 4 (A4):

> The ordered triple of vectors (e, f, g) forms a basis! This happens because the vectors are linearly independent. The basis formed is neither orthogonal nor orthonormal! It is just a basis. For the basis to be orthogonal, the dot product of all vectors, two by two, should be 0. The dot product of vectors (e, f) = 12; therefore, it does not meet the requirements to be an orthogonal basis. No less important, to be orthonormal, the basis should, in addition to being orthogonal, have a norm of 1 for each vector (Student 4).

The application of the Gram-Schmidt Orthogonalization process was presented in detail in the third moment, as follows:

Step 1: Define the origin of the three-dimension coordinate system, including the point O=(0,0,0);

Step 2: Insert an ordered triple of vectors (u_1, u_2, u_3) linearly independent that form a base for the vectorial space V^3;

Step 3: In this phase, apply the Gram-Schmidt Orthogonalization process, that is, insert the triple of vectors (v_1, v_2, v_3) in the GeoGebra 3D Calculator, as given by:

$$v_1=u_1$$

$$v_2=u_2-[(u_2v_1)/|v_1|^2]v_1$$

$$v_3=u_3-[(u_3v_1)/|v_1|^2]v_1-[(u_3v_2)/|v_2|^2]v_2$$

With the following commands:

Vector v_1: v_{1} = u_{1}

Vector v_2: v_{2} = u_{2}-((u_{2} v_{1})/(Length(v_{1})^(2)))v_{1}

Vector v_3: v_{3} = u_{3}-((u_{3} v_{1})/(Length(v_{1})^(2))) v_{1}-((u_{3}v_{2})/(Length(v_{2})^(2))) v_{2}.

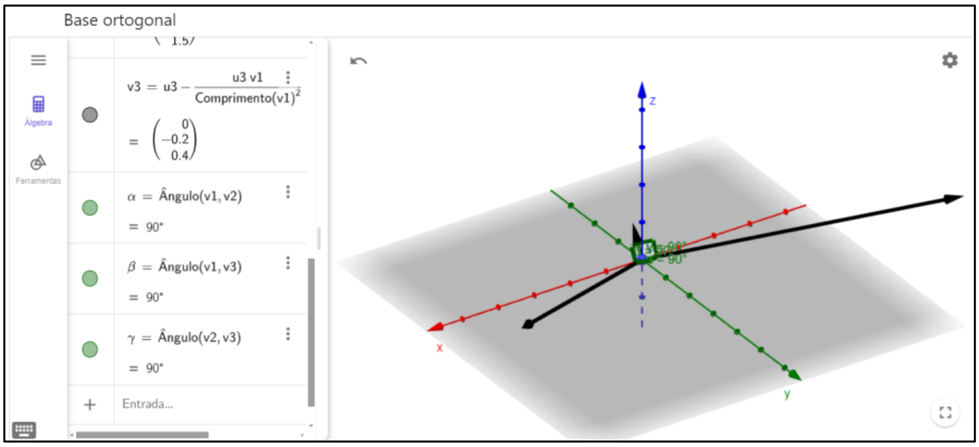
A GeoGebra 3D Calculator window was made available so that students could apply the process from the commands presented, and in the sequence Question 3 was proposed, as presented in Chart 2.

Based on the analysis of answers, 73% of the students obtained an orthogonal base applying the steps described in the process. However, we observed that 58% of the students found it difficult to justify geometrically and/or analytically that the base was orthogonal, as answered by Student 5 (A5): *"The base is really orthogonal since the scalar product between two vectors is equal to 0".* In this answer, we realized that the student did not argument correctly the fact that the base obtained was orthogonal, since for that to be correct, the ordered triple (v_1, v_2, v_3) must form a base for the vectorial space V^3 and the base vectors be orthogonal two by two, that is, the angle between any of two vectors of the base is equal to 90°.

According to Duval's (2003) contributions, such difficulty can be interpreted as an obstacle in the process of integrating different registers of semiotic representation. Although the student uses an algebraic argument accurately, his justification does not evidence the conversion to the geometrical register. As explained by Duval, mathematical understanding requires more than treatment within the same register (as the algebraic), but, above all, the ability to convert between distinct registers, which constitutes one of the main difficulties for students.

We can also highlight Student 6's 6 (A6) answer (Figure 6), who used the GeoGebra 3D Calculator graphic window and geometrical arguments in the resolution of the problem. These included calculations of angles between the base vectors, to verify whether the base was really orthogonal.

Figure 6 - Student 6's GeoGebra 3D Calculator window for question 3.



Source: The authors (2024).

Student A6 complemented the resolution with the following answer: *"After the orthogonalization process, I calculated the angle between vectors and all of them resulted in 90°, hence, the V^3 base became an orthogonal base"*, which indicates that he understood the concepts involved in this question.

Next, we presented the steps to be followed to orthonormalize the orthogonal base previously obtained, that is, to obtain an orthogonal base with unit vectors. This process is known as the Gram-Schmidt orthonormalization process, then, Question 4 was proposed as seen in Chart 2.

Step 1: Repeat the steps used in the application of the Gram-Schmidt orthogonalization process to obtain an orthogonal base for V^3, or insert the orthogonal base (v_1, v_2, v_3) previously obtained;

Step 2: Insert in GeoGebra the triple of vectors (e_1, e_2, e_3), where each vector e_i is the unit vector (versor) of v_i, where $e_i = v_i/|v_i|$ with i=1,2,3.

With the following commands:

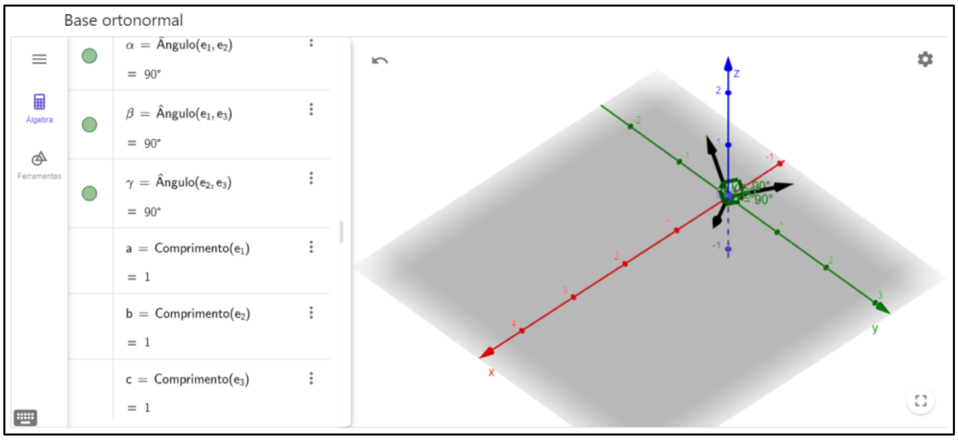
Versor e_1: e_{1}=v_{1} / Length(v_{1});

Versor e_2: e_{2}=v_{2} / Length(v_{2});

Versor e_3: e_{3}=v_{3} / Length(v_{3}).

Student 7 (A7) answered to Question 4 correctly and presented geometrical arguments, as described below and shown in Figure 7: "*By calculating the length, versors and angles, aided by GeoGebra, I found that the orthogonal base that I had obtained previously became an orthonormal base because the length of vectors is equal to 1 and the angle between them is equal to 90°*".

Figure 7 - Student 7's GeoGebra 3D Calculator window for question 4



Source: The authors (2024).

Therefore, we observed that the software helped the student to verify that the triple of vectors (e_1, e_2, e_3) is an orthonormal base orthonormal by calculating the angle between any two vectors and the length of each of the vectors. An orthonormal base contains three orthogonal two by two unit vectors forming a base for the vector space V^3.

On the other hand, Question 4 was not answered by 44% of the participants. We believe that this number is due to the fact that many students could not complete the activity within the time proposed for having difficulties in dealing with the software since they had not mastered the theoretical content that had been explained in previous lessons.

To finish the activity, an essay question was proposed, Question 5, which asked students to share their perceptions about the activity, even if they had not finished answering all previous questions. They should highlight positive and negative aspects and could also make suggestions for the improvement of future activities. We verified that 96% of the answers were positive, with expressions such as: "I liked it a lot, it was fun, congratulations on the initiative, it stimulated knowledge, consolidated the content and made learning easier".

Some students exposed the difficulty they felt using the software and with the subject theoretical content. Others suggested the use of software more often during the lessons. Among the several positive answers, the one given by Student 8 (A8) stood out:

> At first, the contact with the software was strange since this was the first time I had seen it. However, learning analytical geometry with real time visualization of what I was seeking was extremely didactic and rewarding! Surely, it was the best lesson I had in this subject in the entire course. I'm grateful for that (Student 8).

The only negative answer was as follows: "*I thought it was awful, I didn't know how to use the app and was too embarrassed to ask the monitor to help me (: I'd rather do it on paper*".

Visualization has been pointed out as an essential tool in the learning process since it favors the understanding of complex content. Referring to this aspect, Miskulin (1999) highlights two paths that justify the use of computers in teaching.

The first consists in using technology but promoting activities that reproduce traditional teaching. The second path, considered more suitable, involves the exploration of new activities that take advantage of technological resources, enabling new ways for visualization and representation of concepts, mainly in the mathematics field. Therefore, technology might favor new opportunities for the understanding of concepts in an innovative way.

During the development of the activity, we observed the alignment of content to the students' learning needs, since they had already seen the theory of that content. The activity was efficient in reinforcing understanding since most students managed to work autonomously. However, it was also clear that the time planned for the activity (2 hours) was not enough for 9 of the 26 students, who could not answer all the questions proposed within that period.

We also observed, during the development of the activity, the difficulty some students had in executing basic GeoGebra commands. Some asked the professors and monitor to help them. In two moments, the professor in charge used the whiteboard to explain commands that generated doubt. This scenario, also reported in the literature, suggests that the GeoGebra use, despite favoring visualization and concept understanding, requires a period of familiarization by students (Hauenstein, 2022). Studies developed in higher education have shown that the need for teacher support is common in the initial phases of the activities with GeoGebra, especially in contents about straight lines and planes given the abstract character of Analytical Geometry (Silva, 2020).

However, it is worth mentioning that students were interested and engaged during the development of the activity proposed. Thus, as observed in other experiences, students demonstrate interest and active participation throughout the activity, indicating that when the initial difficulties were overcome, the GeoGebra contributed significantly to the dynamic exploration of concepts and to the strengthening of connections between algebraic and geometrical representations (Guadarrama; Becerril, 2023).

Therefore, the GeoGebra use was seen to be particularly advantageous in the teaching of Analytical Geometry for enabling real time visualization of the analytical expressions and geometrical objects by students. Such integration between multiple representations, namely, algebraic, geometrical and computational, favored the understanding of abstract concepts such as linear dependence between vectors, base of the vector space V^3 and the Gram-Schmidt orthogonalization process. The possibility of dynamic handling the figures also stimulated investigations and critical thought, since students could expose their knowledge and explore the content interactively.

FINAL CONSIDERATIONS

Mathematics teaching presents several flaws that need to be addressed. The use of technologies in the classroom is a way of providing students with a different learning environment, where students can develop activities and explore different types of problem solving aided by the visualization of mathematical concepts using software, as demonstrated in this study with GeoGebra.

In this scenario, students were asked to use the GeoGebra window as a tool for calculation and visualization to improve their answers with analytical and geometrical arguments. Students reported having enjoyed the software use, mainly for the benefit of having a geometrical visualization of the theoretical content studied. This shows the scope of this study's objective and confirms the importance of using DT in mathematics teaching and learning.

However, limitations were also identified in the use of that resource since part of the students showed little familiarity with the software, which impacted negatively on the resolution of the questions proposed. This aspect suggests that the inclusion of technological tools in the education context must occur in a continuous and planned manner, so that students can develop greater autonomy in their use. Thus, the use of GeoGebra should be more frequent in these subject lessons and in other mathematical contents.

Regarding the activity focus, we identified satisfactory performance, since 73% of the students obtained an orthogonal base using the Gram-Schmidt orthogonalization process. However, we also observed that 58% of those students, despite reaching a correct answer, had difficulties in justifying their answers using algebraic and geometrical arguments.

Therefore, mathematics teaching with the use of technological resources such as the GeoGebra software was seen to be efficient when addressing the Gram-Schmidt orthogonalization process, thus enabling students to learn Analytical Geometry combined with the visualization of mathematical concepts involved.

VISUALIZAÇÃO DO PROCESSO DE ORTOGONALIZAÇÃO DE GRAM-SCHMIDT COM O GEOGEBRA

RESUMO

Este estudo apresenta o desenvolvimento de uma atividade proposta em sala de aula na disciplina de Geometria Analítica do curso de Bacharelado em Ciências Exatas e Tecnológicas, da Universidade Federal do Recôncavo da Bahia - Campus Cruz das Almas. Na referida atividade, o *software* GeoGebra foi utilizado como ferramenta na investigação da percepção de aprendizagem dos estudantes em relação ao conteúdo de bases e ao processo de ortogonalização de Gram-Schmidt. O objetivo do artigo é apresentar os resultados dessa investigação, analisando como o uso de uma tecnologia digital pode contribuir para a compreensão desses conceitos. Os resultados do estudo revelaram que, apesar de alguns estudantes apresentarem dificuldades com o uso do *software* e com o conteúdo teórico da disciplina, a atividade foi capaz de reforçar a compreensão matemática da maioria dos alunos. Desta maneira, a atividade mostrou-se como uma alternativa para a inclusão de Tecnologias Digitais no ensino de matemática, mais especificamente no ensino de Geometria Analítica.

PALAVRAS-CHAVE: Geometria analítica. Tecnologia digital. Ensino de matemática.

REFERENCES

BORBA, Marcelo de Carvalho. Humans-with-media and continuing education for mathematics teachers in online environments. **ZDM-Mathematics Education** 44(6), p. 801-814, 2012. Available at: https://doi.org/10.1007/s11858-012-0436-8. Access on: Mar. 29th, 2026.

BORBA, Marcelo de Carvalho; VILLARREAL, Mónica E. **Humans-with-media and the reorganization of mathematical thinking**: Information and communication technologies, modeling, visualization and experimentation. Springer Science & Business Media, 2005.

BORBA, Marcelo de Carvalho; SILVA, Ricardo Scucuglia R.; GADANIDIS, George. **Fases das tecnologias digitais em educação matemática:** sala de aula e internet em movimento. Autêntica, 2014.

BORBA, Marcelo de Carvalho; ASKAR, Peter; ENGELBRECHT, Johann; GADANIDIS, George; LLINARES Salvador; AGUILAR, Mario Sánchez. Blended learning, e-learning and mobile learning in mathematics education. **ZDM-Mathematics Education** 48, p. 589–610, 2016. Available at: https://link.springer.com/article/10.1007/s11858-016-0798-4. Access on: Mar. 29th, 2026.

BORBA, Marcelo de Carvalho; SOUTO, Daise Lago Pereira.; CANEDO JR, Neil da Rocha. **Vídeos na educação matemática**: Paulo Freire e a quinta fase das tecnologias digitais. Autêntica, 2022.

BOULOS, Paulo; CAMARGO, Ivan de. **Geometria Analítica**: um tratamento vetorial. São Paulo: Pearson, 2005.

CORRADI, Raquel Polizeli; FRANCO, Valdeni Soliani. Visualização em Geometria, aproximações entre as perspectivas de Duval e Gutiérrez: um estudo com acadêmicos de um curso de licenciatura em Matemática. **Revista BOEM**, Florianópolis, v. 8, n. 16, p. 32-51, 2020. Available at: https://www.revistas.udesc.br/index.php/boem/article/view/17836. Access on: Nov. 26th, 2025.

DUVAL, Raymond. Registros de representações semióticas e funcionamento cognitivo da compreensão em Matemática (1995) *In*: MACHADO, D.A. **Aprendizagem em matemática**: registros de representação semióticas. Campinas: Papirus, 2003. p. 11-33.

DUVAL, Raymond. Figures et visualisation géométrique: «voir» en géométrie. *In*: BAILLÉ, J.; LIMA, J. (ed.). **Du mot au concept**. Figure. Grenoble: Presses Universitaires, 2015. p. 147-182.

FLORES, Cláudia Regina; WAGNER, Débora Regina; BURATTO, Ivone Catarina Freitas. Pesquisa em visualização na educação matemática: conceitos, tendências e perspectivas. **Educação Matemática Pesquisa Revista do Programa de Estudos Pós-Graduados em Educação Matemática**, São Paulo, v. 14, n. 1, p. 31-45, 2012. Available at: https://revistas.pucsp.br/index.php/emp/article/view/8008. Access on: Nov. 21st, 2025.

GEOGEBRA. O que é o GeoGebra?, 2023. Available at: https://www.geogebra.org/about. Access on: Mar. 22nd, 2026.

GIRALDO, Victor; CAETANO, Paulo; MATTOS, Francisco. **Recursos computacionais no ensino de matemática**. Coleção PROFMAT. Sociedade Brasileira de Matemática. Rio de Janeiro, SBM, 2012.

GUADARRAMA, Alberto; BECERRIL, Fernando. Figuras feitas com GeoGebra como estratégia didática para o ensino de geometria analítica. **REAMEC – Revista de Educação em Ciências e Matemática**, v. 11, n. 1, p. e23112, 2023. Available at: https://periodicoscientificos.ufmt.br/ojs/index.php/reamec/article/view/16861. Access on: Mar. 22nd, 2026.

GUEDES, Paulo Cezar Camargo. Aplicação do software geogebra ao ensino da geometria analítica. **Ciência e Natura**, *37*(3), p. 365-375, 2015. Available at: https://doi.org/10.5902/2179460X14555. Access on: May 3rd, 2026.

GUTIÉRREZ, Ángel. **Visualization in 3-Dimensional Geometry: in search of a framework.** University of Valence, Spain, 1996. Available at: https://www.uv.es/Angel.Gutierrez/archivos1/textospdf/Gut96c.pdf. Access on: Mar. 22nd, 2026.

HAUENSTEIN, Débora Marília. **Ensino de geometria analítica auxiliado pela geometria computacional**: uma sequência didática desenvolvida com o uso do GeoGebra. 2022. Dissertation (Mathematics Education Master's Program) — Federal University of Pelotas, Pelotas, 2022. Available at: https://guaiaca.ufpel.edu.br/handle/prefix/8699. Access on: Mar. 22nd, 2026.

MAIA, Lucas Emanuel de Oliveira; GONDIM, Raquel de Souza; VASCONCELOS, Francisco Herbert Lima. Utilização do geogebra para o ensino de geometria: uma revisão sistemática de literatura. **Ensino Da Matemática Em Debate**, *10*(1), p. 31–51, 2023. Available at: https://doi.org/10.23925/2358-4122.2023v10i60031. Access on: Mar. 22nd, 2026.

MISKULIN, Rosana Giaretta Sguerra. **Concepções teórico-metodológicas sobre a introdução e a utilização de computadores no processo ensino/aprendizagem da geometria**. 1999. Thesis (Education Doctorate Program) — State University of Campinas, Campinas, 1999. Available at: https://repositorio.unicamp.br/acervo/detalhe/232997. Access on: Mar. 22nd, 2026.

ORGANIZAÇÃO DAS NAÇÕES UNIDAS PARA A EDUCAÇÃO, A CIÊNCIA E A CULTURA [UNESCO]. **Gestão escolar e tecnologia na educação do Brasil**. 2023**.** Available at: https://www.unesco.org/pt/fieldoffice/brasilia/expertise/educational-management. Access on: Mar. 22nd, 2026.

SILVA, Alessandro Pereira Marcelina da. **O uso do software GeoGebra como recurso metodológico no processo de ensino e aprendizagem de retas e planos**. 2020. End-of-course work (Mathematics Teaching Course) — Federal University of Paraíba, João Pessoa, 2020. Available at: https://repositorio.ufpb.br/jspui/handle/123456789/27697. Access on: Mar. 22nd, 2026.

SOUZA, Leandro Oliveira; LOPES, Celi Espasandin; SOUZA, Antonio Carlos. Os Delineamentos Metodológicos nas Investigações Brasileiras em Educação Estatística. **Revista Perspectivas da Educação Matemática**, UFMS, v. 8, n. 18, p. 506-525, 2015. Available at: https://periodicos.ufms.br/index.php/pedmat/article/view/1461. Access on: Mar. 22nd, 2026.

YOHANNES, Abebayehu; CHEN, Hsiu-Ling. GeoGebra in mathematics education: a systematic review of journal articles published from 2010 to 2020. **Interactive Learning Environments**, p. 5682-5697, 2021. Available at: https://doi.org/10.1080/10494820.2021.2016861. Access on: Mar. 22nd, 2026.

ZIMMERMANN, Walter; CUNNINGHAM, Steve. Editors´ Introduction: What is Mathematical Visualization? *In*: ZIMMERMANN, Walter; CUNNINGHAM, Steve (Eds.). **Visualization in Teaching and Learning Mathematics**. Washington: MAA, 1991. p. 1-7. Available at: https://api.semanticscholar.org/CorpusID:17034582. Access on: Mar. 22nd, 2026.

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