Revista Brasileira de Ensino de Ciência e Tecnologia
ISSN: 1982-873X

https://periodicos.utfpr.edu.br/rbect

How do Undergraduate Mathematics students justify geometric problems based on visual proof?

João Caldato
profjoaocaldato@gmail.com
0000-0001-6951-3590
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Rio de Janeiro, Brasil.

André Pereira da Costa
andre.costa@ufob.edu.br
0000-0003-0303-8656
Universidade Federal do Oeste da Bahia, Barreiras, Bahia, Brasil.

Lilian Nasser
lnasser.mat@gmail.com
0000-0001-6050-4807
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Rio de Janeiro, Brasil.

ABSTRACT

The aim of this article is to investigate how undergraduate students in Mathematics justify geometric problems based on visual proofs. Despite the increase in the number of research works on argumentation and proof in Mathematics teaching, it is observed that activities of this nature are not common in Basic Education. In other words, most teachers who teach Mathematics do not usually ask their students to justify task resolutions or reasoning used. In particular, they do not explore activities that develop visualization in problem solving or justifications for statements of mathematical properties. Visualization constitutes an important component in understanding a concept or a geometric problem, and can contribute to this development. Visualizing a problem means understanding it in terms of a visual (mental) image, constituting an essential part of the solution method. The methodological path used in this qualitative research was the sending of an online questionnaire, elaborated in Google Forms, with three problems characterized by visual proofs for Mathematics undergraduate students from different Higher Education Institutions. Through this collection procedure, the convenience sample included 58 students and the responses were analyzed according to a textual analysis. The data indicate that, when they mobilize visualization, although the undergraduates are able to present a logical chain in an argument, most of them do not realize the need to justify all the arguments used. In addition, in some cases, most participants failed to understand what was the relationship to be demonstrated from the geometric figure, or tried to justify a result different from what was expected. In view of this, it is necessary to encourage the exploration of activities that develop the deductive process with Mathematics undergraduates, in particular, the elaboration and justifications based on visual proofs, so that they use such strategies in their future pedagogical practices, since such proofs can be accessible to Basic Education students.

KEYWORDS: Mathematics undergraduates. Argumentation and Proofs. Geometric Abstractions. Visual Proofs.

1 INTRODUCTION

This article is a revised and expanded version of a research presented at the VIII International Seminar on Research in Mathematics Education (SIPEM), involving the mobilization of undergraduate students in Mathematics about visual proofs, that is, proofs without words, based on the visualization of schemes or figures (CALDATO; PEREIRA DA COSTA; NASSER, 2021).

Such research was motivated by the observation that many Mathematics teachers at the Basic School do not care about developing argumentation skills in their students, nor do they ask them to justify their resolutions for the tasks solved (PEZARINI; MACIEL, 2019). In this way, students lose opportunities to develop their deductive thinking, being restricted only to the resolution of repetitive exercises and the application of algorithms. An explanation for this lack of interest may lie in the initial training of Mathematics teachers. The need to improve the ability of the deductive process in undergraduates has drawn the attention of researchers such as Pietropaolo (2005), Ordem (2015), Mateus (2015), Ferreira (2016) and Nasser and Caldato (2019).

Analyzing the responses of undergraduates to discursive questions of the National Examination of Higher Education (ENADE) (BRASIL, 2008, 2011, 2014), which require deductive reasoning, proposed to undergraduates in the last applications, Nasser and Caldato (2018) investigated whether undergraduate courses in Mathematics have fostered the development of the deductive process of future teachers. In some questions, the answers presented were based on arguments of an empirical nature, only testing the validity of the affirmative for a few examples. Observing the ENADE reports, it was possible to infer that students do not find opportunities in undergraduate Mathematics courses to overcome such difficulties and “[...] they conclude the course with little ability in argumentation, considering the high percentage of answers in white and wrong” (NASSER; CALDATO, 2018, p. 10).

In an attempt to contribute to this issue of teacher training to explore argumentation activities and deductive processes with their future students, this article aims to investigate how undergraduate students in Mathematics justify geometric problems based on visual proofs. The formulated hypothesis is that these subjects cannot deal correctly with situations that involve argumentation and visual proof, which may mean little contact or absence of these situations in Mathematics teaching, as well as in initial teacher training courses.

2 THEORETICAL FOUNDATION

In this text, as a theoretical foundation, research was adopted that discuss proofs and deductive processes, visualization in the teaching of Mathematics and visual proofs, as presented in the next sections.

2.1 Proofs and deductive processes

Realizing the growth in the number of researches and the wide variety of approaches on Proofs in Mathematics teaching, Reid and Knipping (2010) present the main lines of research in this area and discuss works focused on proofs in Mathematics and their various interpretations.

Among such studies, Tall (1995) describes three types of proofs: active, visual and manipulative. While active proofs involve performing a physical action to demonstrate the truth of a result, manipulative proofs involve algebraic simplification. In turn, visual proof involves active elements (and usually has verbal support).

According to Healy and Hoyles (1998, p. 1), “[...] proof is the heart of mathematical thinking, and deductive reasoning, which underpins the process of proving, exemplifies the distinction between mathematics and the empirical sciences”. However, for Martin and Harel (1989 apud HEALY; HOYLES, 2000, p. 396), Mathematics students, in general, are not clear about the distinction between deductive reasoning and empirical or informal reasoning.

For this reason, Caldato (2018) suggests a problematized approach in undergraduate courses, so that the future teacher can establish relationships between mathematical content and its teaching, being able to articulate the deductive processes and demonstrations fostered in academic field with its practice, in order to promote the “whys” of Mathematics in Basic Education.

2.2 Visualization in mathematics teaching

This article addresses the perception of teachers in initial training in relation to visual proofs, or proofs without words, such as those found in the work of Nelsen (1993), which inspired the three problems used in this research.

Shatri and Buza (2017) investigated the use of visualization in teaching and learning for the development of students' critical thinking. According to these researchers, activities with a visual approach favor communication, increase critical thinking and enable an analytical interpretation. In addition, they consider that visualization helps in understanding a concept or a problem. Visualizing a problem means understanding it in terms of a mental visual image, constituting an essential part of the solution method.

Therefore, visualization is a crucial component of learning geometric concepts. In this article, the focus of the investigation is on the mastery of Mathematics undergraduate students in relation to wordless proofs, based only on the visualization of schemes or figures.

In his dissertation, Santos (2014, p. 26) presents a table with the definitions of visualization adopted by 18 national and international researchers, and assumes “[...] the aspect that points to an understanding of visualization as a structuring element in the formation of mental images for the development of visual thinking”. The author analyzes three types of visualization: geometric, algorithmic and contextualized, emphasizing that one does not overlap the other, nor are they equivalent, they only complement each other. However, she points out that the type of visualization most found in the academic literature is the geometric one.

Presmeg (2006) presented a survey of research involving visualization and states that this includes processes of building and transforming a mental visual image and all representations of a spatial nature that may be involved in doing Mathematics.

Visualization cannot be confused with simply seeing with the eyes, but consists of a process that encompasses aspects that go beyond the senses, such as the capacities of: imagination, intuition, understanding and synthesis. Thus, visualization is an important element for abstraction in Mathematics and, consequently, for the development of geometric thinking, whose relevance was recognized by the former National Curricular Parameters (PCN):

> [...] geometric thinking is initially developed through visualization: children know space as something that exists around them. Geometric figures are recognized by their shapes, by their physical appearance, in their entirety, and not by their parts or properties. (BRASIL, 1997, p. 82, emphasis added).

In locating space, the child recognizes geometric figures from the representation of physical objects. It is an abstraction process of an empirical nature, in which the physical interaction with reality plays an essential role. In these geometric activities, “the object is the focus of attention and, only later, the language used for description allows the mind to construct platonic objects, such as 'widthless' lines [...]” (ALMOULOUD, 2017, p 29).

Considering that objects in Mathematics are invisible in our sensible reality and only exist in the intelligible world, these must be analyzed through a representation system (DUVAL, 1995). Thus, mathematical objects are mental constructions that are only understood through our capacity for abstraction.

In the field of Geometry, Pereira da Costa (2019, 2020) argues that geometric abstractions enable the development of geometric thinking. This type of abstraction “[...] is a mental operation, through which we are aware of similarities between our geometric experiences” (PEREIRA DA COSTA, 2020, p. 142). Thus, he proposes a typology of abstractions in Geometry, which consists of an alternative for characterizing geometric thinking, shown in Chart 1.

Chart 1 - Typology of geometric abstractions

Geometric Abstraction	Characterization
Spatial	It is distinguished by the study (or experience) of the concepts of spatial orientation, which also involve location, orientation, displacement, etc.
Perceptual	It is characterized by perceptual and visual sensations. In this geometric abstraction, a figure is analyzed as a whole, devoid of elements and properties.
Analytical	It is marked by the analysis of geometric figures according to their constituent elements and their properties, however, it is not possible to establish inference relationships between these properties.
Descriptive	It is marked by the establishment of implication relations between properties of geometric objects, but without the use of deductive argumentation in the justification of this establishment.
Deductive	Characterized by the study (or experience) of proofs, demonstrations, arguments and conjectures of both an intuitive and deductive nature. Geometry starts to be seen as a mathematical theoretical model, formed by axioms and theorems.
Hypothetical (or theoretical)	It is indicated by the study (or experience) with different geometries, especially the so-called Non-Euclidean Geometries, based on the study of axiomatic theories and the use of an axiomatic formal language.

Source: Pereira da Costa (2019).

In this direction, based on a teaching of Geometry in Basic Education that promotes the development of geometric thinking, it would be advisable that the undergraduate Mathematics courses emphasize proposals anchored in geometric abstractions, in particular, the deductive and hypothetical (theoretical). In this way, the future teacher would have the opportunity to experience arguments and proofs and, consequently, the deductive process, being able to explore this type of activity in their classes, minimizing the performance problems presented by the students.

2.3 Visual proof

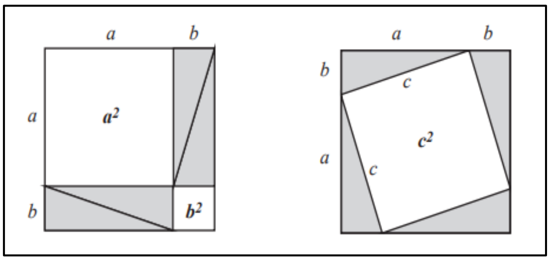
Hanna (2000) questions the intensified use of visualization, that is, if, or to what extent,

> [...] visual representations can be used, not only as evidence for a mathematical statement, but also in its justification. Diagrams and other visual aids have long been used to facilitate understanding, of course. They have been welcomed as heuristic accompaniments to proof, where they can inspire both the theorem to be proved and approaches to the proof itself. [...] Today there is much controversy on this topic, and the question is now being explored by several researchers. (HANNA, 2000, p. 15).

In their research, Borwein and Jörgenson (2001) raise the following question: To what extent can a visual representation be considered a proof?. For the authors, while the mathematical proof traditionally follows a deductive sequence of sentences, a visual proof would be presented as a static image. They point out that such an image may contain the same information as the first, but do not display an explicit way to obtain it, "[...] leaving the viewer to establish what is important (and what is not) and in what order the dependencies should be assessed" (BORWEIN; JÖRGENSON, 2001, p. 899). Therefore, these researchers believe that, in general, visual proof tends to be limited in generalizability. Although the researchers themselves have not definitively answered the introductory question, when problematizing the place of visual representations in Mathematics, they believe that some can be called proofs (HANNA, 2000).

In turn, Reid and Knipping (2010) frame visual proof as a subcategory of generic proofs, which use arguments based on representative examples of a class of objects. In view of this and in order to display a visual proof, the researchers turn to the work of Tall (1995), in particular, the Indian proof for the Pythagorean Theorem, which uses the comparison between the areas of two squares of side $a+b$ to justify the equality $a^2+b^2=c^2$, as shown in Figure 1. This visual proof was presented in the second problem of the questionnaire.

Figure 1 – Visual proof of the Pythagorean Theorem



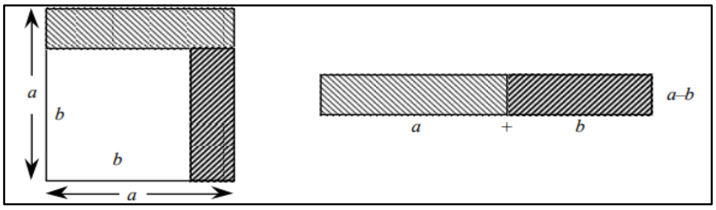
Source: Tall (1995, p. 5).

According to Tall (1995), to understand this visual proof it is essential to imagine the triangles as dynamic objects. Furthermore, it shows that

> [...] any actual drawing will have specific values for a and b, but such a diagram can be seen as a prototype, typical of any right-angled triangle. This gives a kind of proof which is often termed "generic"; it involves "seeing the general in the specific". (TALL, 1995, p. 6).

However, the researcher points out that one of the weak points of the visual proof is due to the limitation of the diagrams, since their application extends only to the class of objects in question. To exemplify this, one can observe Figure 2, in which Tall (1995) uses visual arguments to justify the algebraic identity $a^2 - b^2 = (a + b)(a - b)$, which expresses the difference of the squares of two numbers.

Figure 2 - Visual proof for the difference between two squares



Source: Tall (1995, p. 6).

Note that this proof has certain limitations, as it only applies to positive real numbers, with $a > b$, while the algebraic identity for the difference of two squares holds for any real numbers. In addition, Tall (1995, p. 9) states that "[...] what is satisfactory to an individual at one stage of development may prove to be unsatisfactory later on", that is, the visual proof illustrated in the Figure 2 could be convincing for Basic Education students, for example, but due to its restrictions it could also be questioned by Higher Education students.

3 METHODOLOGICAL PATH

The aim of this research was to investigate how undergraduate students in Mathematics justify geometric problems based on visual proofs. In this sense, a qualitative approach was adopted as it was considered the most appropriate for the study, based on its nature and specificity. For Gil (2008), qualitative research seeks to understand a social phenomenon in its complexity from verbal and visual data collected systematically. In the case of this research, the phenomenon to be understood is constituted by the deductive processes mobilized by undergraduate students in Mathematics in the resolution of problems that explore visual proofs.

To develop the investigation, three problems involving visual proof were proposed, which require the presentation of justification/argumentation of a deductive nature, therefore, they are problems that lie in deductive geometric abstraction, as indicated by Pereira da Costa (2019). Thus, an online questionnaire, prepared in Google Forms, was used as a data collection instrument. Subsequently, the link (https://forms.gle/BpTUxTKniNkeNapq6) was shared in groups on social networks or sent by email to different Higher Education Institutions, and the sample was selected for convenience. This form of data collection allowed a greater scope of the sample, especially in the geographical sense.

In all, 60 participants answered the questionnaire during the month of April 2021. However, two of them did not authorize the use of the data. In view of this, the sample consisted of 58 undergraduates in Mathematics, designated, respectively, by L1, L2, ..., L58. The students came from seven Brazilian states (RJ, BA, PB, PE, RN, TO and SP), whose year of admission to graduation varied between 2009 and 2021. However, as the purpose of this article is not to evaluate institutions, their names have been preserved.

The collection procedure can be classified, despite the limitations, as a survey research, described as “obtaining data or information about characteristics, actions or opinions of a certain group of people, indicated as representative of a target population [...] through a research tool, usually a questionnaire” (ALYRIO, 2009, p. 129).

Thus, as an analytical tool, the choice was to use a textual analysis of the data, through which it was possible to elaborate a simple classification of the students' responses/productions to the three problems contained in the questionnaire, as shown in Chart 2.

Chart 2 - Types of responses and data analysis criteria

Answer types	**Description**
Identification of the mathematical relationship with complete justification	The participant identified the mathematical relationship addressed in the problem and correctly justified all the arguments used.
Identification of the mathematical relationship with incomplete justification	The participant identified the mathematical relationship addressed in the problem and partially justified the arguments used.
Identification of the mathematical relationship without justification	The participant identified the mathematical relationship addressed in the problem and did not present valid arguments.
No identification of mathematical relationship	The participant did not correctly identify the mathematical relationship addressed in the problem.
No response	The participant did not respond to the problem, that is, he said he did not know or left it blank.

Source: Prepared by the authors (2022).

4 DATA ANALYSIS

This article presents and discusses the responses of undergraduates to the three problems described in the questionnaire, which are illustrated in Figures 3, 5 and 8.

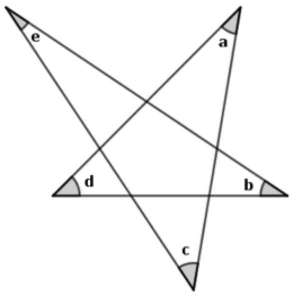
The first problem (Figure 3) addresses the measurement of the internal angles of a pentagonal star, also known as a pentagram or five-pointed star. In the justification, if the participant manages to prove that the sum of the measures of the internal angles measures 180°, he will certainly be mobilizing the deductive geometric abstraction proposed by Pereira da Costa (2019), since he will be able to perceive (Euclidean) Geometry as a mathematical model consisting of a set of theorems and axioms.

Figure 3 – Problem 1

The figure on the side can be used to justify a mathematical result. Based on this answer:

a) What would this result be?

b) How did you come to this conclusion? **Justify your answer.**



Source: Caldato, Pereira da Costa, Nasser (2021, p. 692).

There are at least five correct ways to solve this problem. But, basically, to find out that the sum of the measures of the interior angles is equal to 180° ($a + b + c + d + e = 180°$), it is necessary to make use of two properties linked to Euclidean triangles: (i) the sum of the measures of the interior angles of any triangle is equal to a 180°; (ii) in any triangle, the value of the measure of an exterior angle is equal to the sum of the measures of the two interior angles not adjacent to it. This last relation is also known as the exterior angle theorem. The difficulty of this item lies in the lack of understanding of these properties about Euclidean triangles.

Nelsen (1993) when proposing this problem in his book, suggests visual aids by means of rays, from one of the vertices, parallel to two sides of the star. The use of auxiliary elements goes in the direction of what was described by Hanna (2000), as it helps in understanding the problem and leads to the construction of a justification for the result.

From data analysis, the following types of responses described in Chart 3 were identified:

Chart 3 – Types of answers given by undergraduate students to problem 1

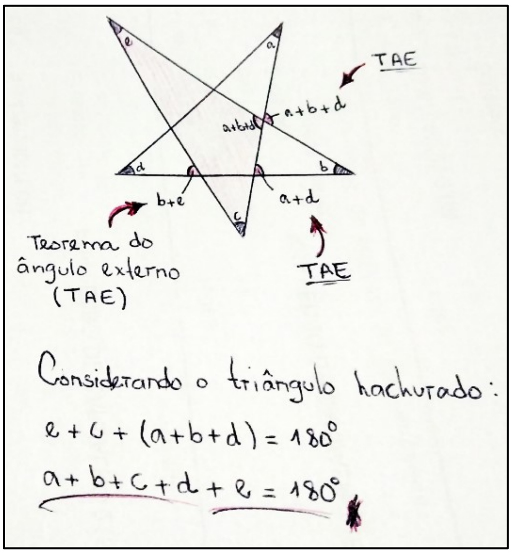
Types of responses	Description	Frequency
Identification of the mathematical relationship with complete justification	The participant identified the mathematical relationship and justified it correctly using the external angle theorem and based on the sum of the measures of the internal angles.	12
Identification of the mathematical relationship with incomplete justification	The participant identified the mathematical relationship and partially justified it using the external angle theorem and based on the sum of the measures of the internal angles.	04
Identification of the mathematical relationship without justification	The participant identified the mathematical relationship and did not justify it using the external angle theorem or using the sum of the measures of the internal angles.	02
Mathematical relationship not identified	The participant did not correctly identify the mathematical relationship.	37

No Response	The participant has not responded to the issue.	03

Source: Prepared by the authors (2022).

Twelve students presented correct justifications for the problem, referring to the two properties of Euclidean triangles, that is, they made use of the theorem of the external angle and the sum of the internal angles, as can be seen in Figure 4 (Response from L14). Therefore, this data indicates that approximately 20% of the undergraduates used deductive geometric abstraction, as previously mentioned.

Figure 4 - Answer with full identification and justification



Source: Caldato, Pereira da Costa, Nasser (2021, p. 693).

In addition, four participants forwarded a correct answer. However, they did not present an adequate justification, such as the L20 response; and two had an incomplete answer, justifying another result, that is, another mathematical relationship, such as the L30 answer. Thus, these results seem to indicate that such students have not yet achieved deductive geometric abstraction. The following fragments corroborate this fact:

> The sum of the 5 angles is 180 degrees, I reached this conclusion by moving the angles and placing them side by side. (Response from L20).

> The sum of the opening measure of the pentagon angles is 540°, using the triangles that make up the figure. (Response from L30).

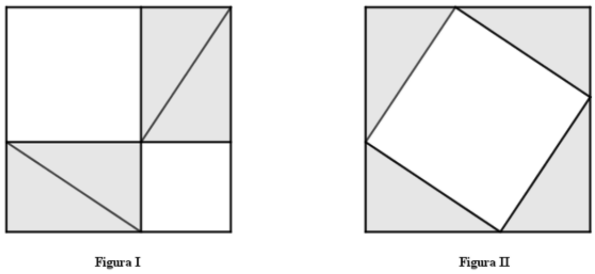
It is noted that 63.79% of the participants were unable to identify the mathematical relationship explored in the problem; and 5.17% did not answer the problem. These data seem to show that these students do not work in deductive geometric abstraction, that is, they cannot carry out proofs and demonstrations in Geometry, nor do they use argumentation.

In addition, the data described in Table 3 indicate that the Mathematics undergraduates in this sample do not observe the importance and the need to justify the reasoning presented in solving a problem. Such behavior can have direct implications in Basic Education, since the justification for solving mathematical problems is not worked with students at this school level, especially those of a geometric nature.

The second problem concerns the Pythagorean Theorem. Initially, figures I and II were presented, which can be used to justify this theorem in the context of Basic Education, as shown in Figure 5:

Figure 5 - Problem 2

Figures I and II represent two squares that can be used to justify a mathematical relationship that is taught in Basic Education.



Based on this answer:

a) What would this mathematical relationship be?

b) Could you justify this result using the figures above? **If yes, how? If not, why not?**

Source: Prepared by the authors (2021).

The objective of item (a) was to investigate whether the undergraduates were aware that such figures were related to a visual proof of the Pythagorean Theorem. Based on this, the idea of item (b) was to present a justification for this visual proof and for that it was necessary: (i) to compare the areas of figures I and II and argue that they are equal; (ii) demonstrate that the quadrilateral inscribed in figure II is a square, from the congruence of triangles and the measures of the angles. In this problem, in line with Tall (1995), imagining the triangles as dynamic objects favors the visualization of the equivalence between the areas of the squares and the construction of a generic justification for the mathematical relationship.

Data analysis showed that only 34.48% of the sample managed to identify the mathematical relationship in item (a), as shown in Chart 4:

Chart 4 - Types of answers given by undergraduates to problem 2(a)

Types of responses	**Description**	**Frequency**
Identification of the mathematical relationship	The answer classified in this typology explicitly mentioned that it was the Pythagorean Theorem.	20
No identification of the mathematical relation	The answer classified in this typology mentioned that it was a relation about areas or plane figures or plane geometry, but it did not make explicit that it was the Pythagorean Theorem	10
	The answer classified in this typology mentioned the use of the Pythagorean Theorem in calculating the area of plane figures	03
	The answer classified in this typology mentioned the notable products.	05

	The answer classified in this typology mentioned other Mathematics topics, such as metric relations and quadratic equations.	06
No response	7 participants said they did not know and 8 left blank.	15

Source: Prepared by the authors (2022).

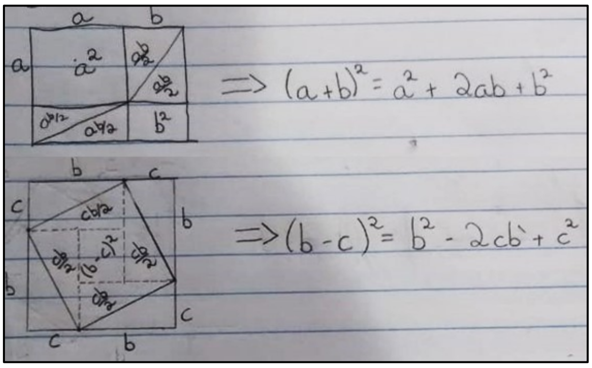
Among the 20 participants who mentioned that the relation was the Pythagorean Theorem, mobilizing the deductive geometric abstraction, it is worth highlighting the response of L52 who only wrote the following expression: $c^2 = a^2 + b^2$. Although the answer is correct, considering that it mentions the theorem, it is important to point out that this relationship is not restricted to an algebraic expression, even because the problem did not initially name the sides of the figures. And if the measurement of the length of the hypotenuse had been associated with a letter other than c, would the relationship remain the same? Questions like this need to be problematized in Mathematics teaching.

One type of answer that deserves to be highlighted, even though it was considered incorrect, was mentioning the application of the Pythagorean Theorem in the numerical calculation related to the area of plane figures. This idea was present in the response of three undergraduate students and one of them was illustrated in the following fragment:

> It is related to the quantity "area" which is a characteristic of a geometric object, in addition to the calculation involving length and may or may not use the Pythagorean theorem as it involves representations of right triangles, to solve a given proposed problem, in addition to being related to the completing the square method as in Figure I. (Response from L33).

Although the problem on screen brought in the statement that both squares were associated with a mathematical relationship, some participants sought to find two relationships. Among them, six students did not identify the Pythagorean Theorem and mentioned the notable products, which was the case of L57, who related the first figure to the square of the sum of two terms and the second figure to the square of the difference of two terms, as shows Figure 6.

Figure 6 – Unidentified response to the Pythagorean Theorem



Source: Survey data (2021).

It is important to highlight that, in fact, the first represented geometric figure could be used to geometrically prove the square of the sum of two terms, in the context of Basic Education, as discussed in the theoretical foundation based on Tall

(1995). However, note that the justification for the difference of two terms is incorrect, since, among other reasons, it is not possible to observe squares of areas b^2 and c^2..

Regarding item (b) of the second problem, in which students should justify the visual proof of the Pythagorean Theorem, mobilizing deductive geometric abstraction, Chart 5 presents the types of responses that were identified in the questionnaires:

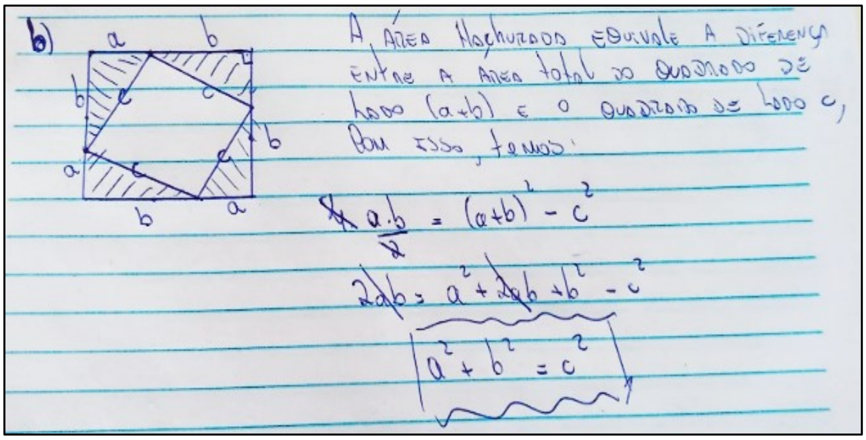
Chart 5 - Types of answers given by undergraduates to problem 2(b)

Types of responses	**Description**	**Frequency**
Identification of the mathematical relationship with complete justification	The participant identified the theorem in item (a) and correctly justified all the arguments used in item (b).	00
Identification of the mathematical relationship with incomplete justification	The participant identified in item (a) the theorem and partially justified the arguments used in item (b). In this case, the answer classified in this typology either did not justify that the quadrilateral inscribed in figure II is a square or stated that it was by comparing the areas of the squares, but without details of the reasoning.	08
Identification of the mathematical relationship without justification	The participant identified the theorem in item (a) and did not present valid arguments. In this case, the answer classified in this typology, either showed an incorrect argument, or did not present a justification.	12
No identification of mathematical relationship	The participant did not identify the theorem in item (a) and responded to item (b). In this case, the response classified in this typology identified another relationship in item (a).	23
No response	The participant did not identify the theorem in item (a) and did not respond to item (b)	15

Source: Prepared by the authors (2022).

Note that none of the 20 participants, who identified in part (a) that the problem was about the Pythagorean Theorem, managed to describe a completely correct justification of the visual proof. Among the undergraduates who identified the mathematical relationship, eight presented incomplete justifications in item (b). The most common mistake in this typology, observed in five responses, was not arguing that the quadrilateral inscribed in figure II actually represents a square, so that it would be possible to calculate the measure of the area as being the measure of the length of the side raised to the second power. Figure 7 illustrates this type of response, which was given by L56:

Figure 7 - Answer with identification and incomplete justification



Source: Survey data (2021).

Among the twelve undergraduates who identified the mathematical relationship, but did not present valid arguments, the answer given by L16 called our attention: “If we have the data of the triangles or the square, through the Pythagorean theorem we can find the other measures”.

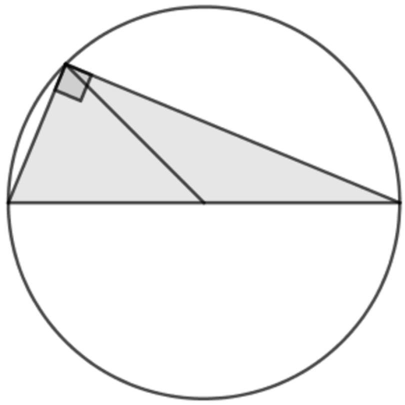
Although the student has correctly answered item (a), in order to justify the Pythagorean Theorem, he suggests using the theorem itself to determine the lengths of some sides. This is an indication that the future teacher does not have the clarity (or certainty) of the meaning of a demonstration. This corroborates the point made by Martin and Harel (1989 apud HEALY; HOYLES, 2000).

The results of problem 2 indicate that most participants do not recognize one of the visual proofs for the Pythagorean Theorem and do not know how to justify it. Some research shows that future teachers, in general, do not master strategies to validate this theorem (CALDATO, 2018; MATEUS, 2015). Such indications go in the opposite direction to the guidelines described in the National Common Curricular Base (BNCC) (BRASIL, 2018), which prescribes that in the 9th year of Elementary School, the student must already be able to demonstrate it, at least, through the similarity of triangles. Therefore, this may be further proof that Mathematics teaching has prioritized the use of “ready” results, to the detriment of understanding the concepts.

The third problem (Figure 8) involves a property of the right triangle that can be easily justified if the triangle is inscribed in a circle, thus mobilizing deductive geometric abstraction.

Figure 8 – Problem 3

The figure below represents a right triangle inscribed in a circle and can be used to justify that the median relative to the hypotenuse measures half of the hypotenuse. **Argue how this is possible.**



Source: Caldato, Pereira da Costa, Nasser (2021, p. 694).

To justify the property presented in the third problem it was necessary: (i) to realize that the hypotenuse of the triangle coincides with the diameter of the circumference, and to justify this fact; (ii) observe that the median coincides with a radius and, therefore, its measure is half the length of the hypotenuse.

The difficulty of this item lies in the justification that a right triangle is always inscribed in a semicircle. As the measure of an angle inscribed in a circle corresponds to half the length of the arc subtended by its sides, an angle of 90° subtends an arc of 180° and, therefore, the hypotenuse coincides with a diameter of the circle. As described by Presmeg (2006), visualization should not be confused with simply seeing with the eyes. And in this sense, one conjecture was that many participants would look at the proposed figure without a critical posture, inducing them not to consider it necessary to argue the fact that the hypotenuse coincides with the diameter.

From the analysis of the responses presented to problem 3, the following types of responses described in Chart 6 were identified:

Chart 6 - Types of answers given by undergraduates to problem 3

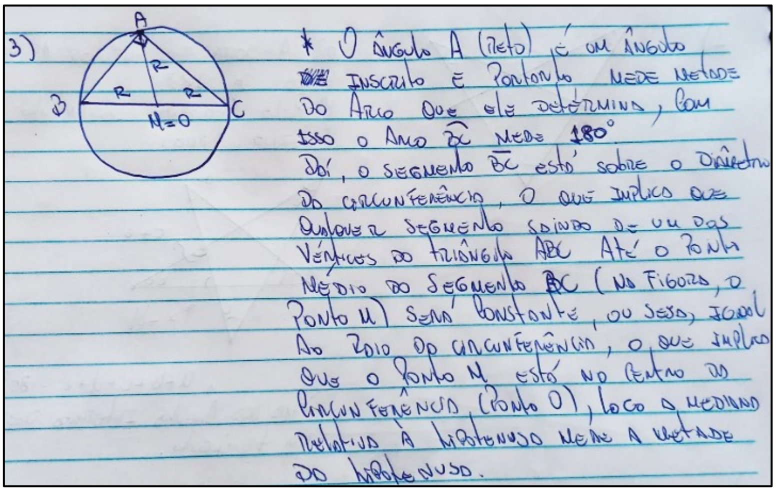
Types of responses	Description	Frequency
Identification of the mathematical relationship with complete justification	The participant correctly justified that an inscribed angle of 90° subtends an arc of 180°, therefore the hypotenuse coincides with the diameter, concluding that the median measures half of the hypotenuse.	06
Identification of mathematical relationship with incomplete justification	The participant did not justify correctly because the hypotenuse coincides with the diameter. Even so, he concluded that the median measures half the hypotenuse.	19
Identification of the mathematical relationship without justification	The participant did not present valid arguments or just repeated the statement.	10

No identification of mathematical relationship	All participants identified the mathematical relationship, which was in the statement.	00
No response	12 participants said they did not know and 11 left blank.	23

Source: Prepared by the authors (2022).

Only six undergraduates correctly justified the fact that the hypotenuse coincides with the diameter of the circumference, demonstrating that they act in deductive geometric abstraction, as exemplified in Figure 9 (Response from L56) and concluded the question with an adequate argument.

Figure 9 - Response with identification and complete justification



Source: Caldato, Pereira da Costa, Nasser (2021, p. 695).

On the other hand, 32.75% of the participants forwarded the reasoning correctly, but did not feel the need to justify why the hypotenuse coincides with the diameter. The following fragment corroborates this fact:

> Note that the hypotenuse of the inscribed triangle measures 2r (2×radius). On the other hand, the median relative to the hypotenuse measures r (radius). So we can say that the median is half the hypotenuse. (Response from L12).

The rest of the students did not respond or used inappropriate arguments, not being able to justify the property, therefore, they did not act in the deductive geometric abstraction. This behavior is in line with the hypothesis formulated in this research and confirms that it is necessary to reinforce with undergraduates the need to justify all the steps used in deductive reasoning, so that they get used to using this strategy with their future students. In this sense, visualization helps in the argumentation process, as it has a positive effect on the performance and development of students' critical thinking (SHATRI; BUZA, 2017).

5 FINAL CONSIDERATIONS

This research investigated how undergraduate students in Mathematics justify geometric problems based on visual proofs. Based on the responses, it was found that, in problem 1, about 80% of the participants were unsuccessful in their justifications regarding the sum of the measures of the internal angles of a pentagonal star. Furthermore, in problem 2, no student was able to identify the mathematical relationship and justify all the arguments used in the visual proof of the Pythagorean Theorem. It is also noteworthy that only about 20% of future teachers associated geometric representations with a proof of the theorem. In problem 3, only six undergraduates correctly justified that the median of a right triangle measures half of the hypotenuse. On the other hand, approximately one third of the sample did not find it necessary to justify that the hypotenuse of the triangle coincided with the diameter of the circumference, despite presenting coherent reasoning.

In general, considering the responses to the three problems, there is an indication that the deductive geometric abstraction proposed by Pereira da Costa (2019, 2020) would not be being explored in undergraduate courses in Mathematics, which may have effects on the way in which Geometry will be taught by these future teachers in Basic Education. Such abstraction is marked by experience with arguments, conjectures, proofs and demonstrations of both an intuitive and deductive nature.

In view of this, the argumentative ability must always be encouraged by the teacher, whether in Higher Education or Basic Education, asking the student to justify their resolution strategies for the proposed problems. After all, mastery of the deductive process must be built throughout the student's trajectory. In addition, because they are not characterized by a formal mathematical language, visual proofs tend to be accessible to Basic Education students. And, by stimulating students' imagination and abstraction, such proofs can contribute to the development of argumentative reasoning during their training.

COMO ESTUDANTES DE LICENCIATURA EM MATEMÁTICA JUSTIFICAM PROBLEMAS GEOMÉTRICOS A PARTIR DE PROVAS VISUAIS?

RESUMO

O objetivo deste artigo é investigar como estudantes de Licenciatura em Matemática justificam problemas geométricos a partir de provas visuais. Apesar do aumento no número de trabalhos de pesquisa em argumentação e provas no ensino de Matemática, observa-se que atividades desta natureza não são comuns na Educação Básica. Em outras palavras, grande parte dos professores que ensina Matemática não costuma pedir que seus alunos justifiquem as resoluções de tarefas ou os raciocínios utilizados. Em particular, não exploram atividades que desenvolvam a visualização na resolução de problemas ou justificativas para afirmativas de propriedades matemáticas. A visualização constitui um componente importante na compreensão de um conceito ou de um problema geométrico, e pode contribuir para esse desenvolvimento. Visualizar um problema significa compreendê-lo em termos de uma imagem visual (mental), constituindo uma parte essencial do método de solução. O percurso metodológico utilizado nesta pesquisa qualitativa foi o envio de um questionário online, elaborado no Google Forms, com três problemas caracterizados por provas visuais para licenciandos de Matemática de diversas Instituições de Ensino Superior. Por meio deste procedimento de coleta, a amostra por conveniência abrangeu 58 estudantes e as respostas foram analisadas de acordo com uma análise textual. Os dados indicam que, ao mobilizarem a visualização, embora os licenciandos consigam apresentar um encadeamento lógico em uma argumentação, grande parte deles não percebe a necessidade de justificar todos os argumentos utilizados. Além disso, em alguns casos, a maioria dos participantes não conseguiu perceber qual era a relação a ser demonstrada a partir da figura geométrica, ou procurou justificar um resultado distinto do que era esperado. Em vista disso, é preciso incentivar a exploração de atividades que desenvolvam o processo dedutivo com os licenciandos de Matemática, em especial, a elaboração e justificativas a partir de provas visuais, a fim de que eles utilizem tais estratégias em suas práticas pedagógicas futuras, uma vez que tais provas podem ser acessíveis aos estudantes da Educação Básica.

PALAVRAS-CHAVE: Licenciandos de Matemática. Argumentação e Provas. Abstrações Geométricas. Provas Visuais.

BIBLIOGRAPHIC REFERENCES

ALMOULOUD, S. A. Fundamentos norteadores das teorias da Educação Matemática: perspectivas e diversidade. **Amazônia**, Belém, v. 13, n. 27, p. 5-35, 2017.Available at: https://periodicos.ufpa.br/index.php/revistaamazonia/article/view/5514. Accessed on: Nov. 14th, 2022.

ALYRIO, R. D. **Métodos e técnicas de pesquisa em administração**. Volume único. Rio de Janeiro: Fundação CECIERJ, 2009.

BORWEIN, P.; JÖRGENSON, L. Visible Structures in Number Theory. **The Mathematical Association of America**, v. 108, n. 10, p. 897-910, dez. 2001. Available at: https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/Borwein897-910.pdf. Accessed on: Nov. 14th, 2022.

BRASIL. **Base Nacional Comum Curricular**: educação é a base. Brasília, MEC/SEB, 2018. Available at: http://basenacionalcomum.mec.gov.br. Accessed on: Nov. 14th, 2022.

BRASIL. **Relatórios do ENADE**: 2014. Brasília: MEC/INEP. Available at: https://download.inep.gov.br/educacao_superior/enade/relatorio_sintese/2014/2014_rel_matematica.pdf. Accessed on: Nov. 14th, 2022.

BRASIL. **Relatórios do ENADE**: 2011. Brasília: MEC/INEP. Available at: https://download.inep.gov.br/educacao_superior/enade/relatorio_sintese/2011/2011_rel_matematica.pdf. Accessed on: Nov. 14th, 2022.

BRASIL. **Relatórios do ENADE**: 2008. Brasília: MEC/INEP. Available at: https://download.inep.gov.br/educacao_superior/enade/relatorio_sintese/2008/2008_rel_sint_matematica.pdf. Accessed on: Nov. 14th, 2022.

BRASIL. **Parâmetros Curriculares Nacionais**: Matemática. Secretaria de Educação Fundamental. Brasília: MEC/SEF, 1997. Available at: http://portal.mec.gov.br/seb/arquivos/pdf/livro03.pdf. Accessed on: Nov. 14th, 2022.

CALDATO, J. C.; PEREIRA DA COSTA, A.; NASSER, L. Analisando a Interpretação de Provas Visuais por Licenciandos de Matemática. *In*: SEMINÁRIO INTERNACIONAL DE PESQUISA EM EDUCAÇÃO MATEMÁTICA, 8, 2021, Uberlândia. **Anais** [...]. Uberlândia: SBEM, 2021, p. 684-698. Disponível em: http://www.sbembrasil.org.br/files/sipemviii.pdf. Accessed on: Nov. 14th, 2022.

CALDATO, J. C. **Argumentação, prova e demonstração**: uma investigação sobre as concepções de ingressantes no curso de licenciatura em Matemática. 219 f. Dissertação (Mestrado em Ensino de Matemática) – Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2018. Available at: http://www.pg.im.ufrj.br/pemat/MSc%2090_Carlos%20Caldato%20Correia.pdf. Accessed on: Nov. 14th, 2022.

DUVAL, R. **Sémiosis et pensée humaine**: registres sémiotiques et apprentissages intellectuels. Berne: Peter Lang, 1995.

FERREIRA, M. B. C. **Uma organização didática em quadrilátero que aproxime o aluno de licenciatura das demonstrações geométricas**. 342 f. Tese (Doutorado em Educação Matemática) – Pontifícia Universidade Católica de São Paulo, São Paulo, 2016. Available at: https://tede2.pucsp.br/handle/handle/18952. Accessed on: Nov. 14th, 2022.

GIL, A. C. **Métodos e técnicas de pesquisa social**. 6. ed. São Paulo: Atlas, 2008.

HANNA, G. Proof, explanation and exploration: an overview. **Educational Studies in Mathematics**, v. 44, n. 1, p. 5-23, 2000. Available at: https://www.researchgate.net/publication/226598348_Proof_Explanation_and_Exploration_An_Overview. Accessed on: Nov. 14th, 2022.

HEALY, L.; HOYLES, C. **Justifying and Proving in School Mathematics**: Technical report on the nationwide survey. London: Institute of Education, University of London, 1998. 120 p.

HEALY, L.; HOYLES, C. A Study of Proof Conceptions in Algebra**. Journal for Research in Mathematics Education**, v. 31, n. 4, p. 396-428, jul. 2000. Available at: https://www.jstor.org/stable/749651. Accessed on: Nov. 14th, 2022.

MATEUS, M. E. A. **Um estudo sobre os conhecimentos necessários ao professor de Matemática para a exploração de noções concernentes às demonstrações e provas na Educação Básica**. 269 f. Tese (Doutorado em Educação Matemática) – Universidade Anhanguera de São Paulo, São Paulo, 2015. Available at: https://repositorio.pgsskroton.com//handle/123456789/3489. Accessed on: Nov. 14th, 2022.

NASSER, L.; CALDATO, J. O desenvolvimento do processo dedutivo nos cursos de Licenciatura em Matemática. In: SEMINÁRIO INTERNACIONAL DE PESQUISA EM EDUCAÇÃO MATEMÁTICA, 7, 2018, Foz do Iguaçu. **Anais** [...]. Foz do Iguaçu: SIPEM, 2018, p. 1-12. Available at: http://www.sbemparana.com.br/eventos/index.php/SIPEM/VII_SIPEM/paper/view/436/232. Accessed on: Nov. 14th, 2022.

NASSER, L.; CALDATO, J. Investigação sobre o desenvolvimento do processo dedutivo nos cursos de Licenciatura em Matemática. **REnCiMa**, São Paulo, v. 10, n. 2, p. 80-96, 2019. Available at: https://revistapos.cruzeirodosul.edu.br/index.php/rencima/article/view/2333. Accessed on: Nov. 14th, 2022.

NELSEN, R. B. **Proofs without words**: exercises in visual thinking. 1 ed. USA: The Mathematical Association of America, 1993.

ORDEM, J. **Prova e demonstração em geometria plana**: concepções de estudantes da licenciatura em ensino de Matemática em Moçambique. 341 f. Tese (Doutorado em Educação Matemática) – Pontifícia Universidade Católica de São Paulo, São Paulo, 2015. Available at: https://tede2.pucsp.br/handle/handle/11035. Accessed on: Nov. 14th, 2022.

PEREIRA DA COSTA, A. **A construção de um modelo de níveis de desenvolvimento do pensamento geométrico**: o caso dos quadriláteros notáveis. 402 f. Tese (Doutorado em Educação Matemática e Tecnológica) - Universidade Federal de Pernambuco, Recife, 2019. Available at: https://repositorio.ufpe.br/handle/123456789/33431. Accessed on: Nov. 14th, 2022.

PEREIRA DA COSTA, A. Abstrações em Geometria: uma alternativa para análise do pensamento geométrico. **Vidya**, Santa Maria, v. 40, n. 1, p. 1-21, jan./jun. 2020. Available at: https://periodicos.ufn.edu.br/index.php/VIDYA/article/view/2996/2528. Accessed on: Nov. 14th, 2022.

PEZARINI, A. R.; MACIEL, M. D. Avaliação dos argumentos e das argumentações produzidas pelos estudantes de Ciências e Biologia a partir de uma proposta didática pautada em Toulmin e Bonini. **REnCiMa**, São Paulo, v. 10, n. 1, p. 27-47, 2019. Available at: https://revistapos.cruzeirodosul.edu.br/index.php/rencima/article/view/2251. Accessed on: Nov. 14th, 2022.

PIETROPAOLO, R. C. **(Re)significar a demonstração nos currículos da educação básica e da formação de professores de Matemática**. 388 f. Tese (Doutorado em Educação Matemática) – Pontifícia Universidade Católica de São Paulo, São Paulo, 2005. Available at: https://sapientia.pucsp.br/handle/handle/11074. Accessed on: Nov. 14th, 2022.

PRESMEG, N. Research on Visualization in learning and teaching mathematics. In: GUTIÉRREZ, A.; BOERO, P. (Org.). **Handbook of Research on the Psychology of Mathematics Education**: Past, Present and Future. Rotterdam: Sense Publishers, 2006. p. 205-236.

REID, D. A.; KNIPPING, C. **Proof in Mathematics Education**: Research, Learning and Teaching. Rotterdam: Sense Publishers, 2010.

SANTOS, A. H. **Um estudo epistemológico da visualização matemática**: o acesso ao conhecimento matemático no ensino por intermédio dos processos de visualização. 98 f. Dissertação (Mestrado em Educação Matemática) – Universidade Federal do Paraná, Curitiba, 2014. Available at: https://acervodigital.ufpr.br/handle/1884/37264. Accessed on: Nov. 14th, 2022.

SHATRI, K.; BUZA. K. The Use of Visualization in Teaching and Learning Process for Developing Critical Thinking of Students. **European Journal of Social Sciences Education and Research**, v. 4, n. 1, p. 71-74, jan./abr. 2017. Available at: https://revistia.com/index.php/ejser/article/view/6470. Accessed on: Nov. 14th, 2022.

TALL, D. Cognitive development, representations and proof. In: **Proceedings of justifying and proving in school mathematics**. London: Institute of Education, 1995. p. 27-38.

Received: Jun. 30th, 2022.
Approved: Nov. 14th, 2022.
DOI: 10.3895/rbect.v15n3.15688
How to cite: CALDATO, J.; COSTA, A.; NASSER, L. How do Undergraduate Mathematics students justify geometric problems from visual proofs?. Brazilian Journal of Science Teaching and Technology, Ponta Grossa, Special Edition, p. 1-21, Dec. 2022. Available at: <https://periodicos.utfpr.edu.br/rbect/article/view/15688>. Access on: XXX.
Mailing address: João Caldato - profjoaocaldato@gmail.com