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# Mathematical Research in the midst of the theoretical framework of the Three Worlds of Mathematics in the Mathematics teacher training

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## ABSTRACT

In this work, the objective is to discuss the consonance between the process of Mathematical Research in the classroom and the theoretical framework of the Three Worlds of Mathematics (Embodied, Symbolic and Formal) in the midst of the initial training of Mathematics teachers. This is a bibliographic research, in which, through textual analysis (books, theses, dissertations, articles, among others) the theoretical encounter between two or more fields of study is promoted. Therefore, hypothetically, the resolution of second degree equations is debated, through its specific formula, having as possible interlocutors, academics of a Mathematics Degree course, within the scope of the discipline of Elementary School Practices. Then, dialogues are developed in the methodological perspective of Mathematical Research, with a view to involving the three different mathematical aspects of acting on mathematical objects (practical, theoretical and formal) and, intertwined with the three levels of abstraction of mathematical knowledge (structural, operational and formal). The analyzes of the discussions, based on the theoretical framework adopted, allow us to conclude that the insertion of the practice of Mathematical Research in consonance with the theoretical framework of the Three Worlds of Mathematics allows the teacher to research, plan, act mathematically and analyze conjectures of resolutions of one or more problems, as well as fostering and validating the discussions and resolutions presented by its students in the midst of the set of areas of mathematical knowledge (Arithmetic, Geometry, Algebra, Analytical Geometry, among others), in order to highlight particularities and/or highlight the similarities and conjunctions between them.

**KEYWORDS:** Mathematical Education. Mathematical Research. Three Worlds of Mathematics.

## INTRODUCTION

In this work, the aim is to discuss the consonance between the methodology of Mathematical Research in the classroom and the theoretical framework of the Three Worlds of Mathematics (Embodied, Symbolic and Formal), in the midst of the initial training of Mathematics teachers. Thus, possible hypothetical discussions are presented between the teacher educator and his undergraduate students in Mathematics, in the scenario of the subject of Elementary School Practices and, having as object of study, the process of teaching and learning the resolution of second-degree equations by through its specific formula. It is noteworthy that this article corresponds to an expansion of the work by Polegatti and Savioli (2021b), presented at the VIII International Seminar on Research in Mathematics Education.

It is a bibliographical research that "uses data or categories already worked on by other researchers and properly registered. The texts become sources of the themes to be researched" (SEVERINO, 2015, p. 122). In the dynamics of bibliographic research, the theoretical framework is inherent to the investigative process with interaction between the researcher and the authors referenced through the analysis and possible relationships of their documents or published works (books, theses, dissertations, articles, among others).

It is agreed that "[...] bibliographical research is not a mere repetition of what has already been said or written about a certain subject, but allows the examination of a theme under a new focus or approach, reaching innovative conclusions" (MARCONI; LAKATOS, 2010, p. 166). Thus, this work proposes the insertion of the methodology of Mathematical Research in the scope of the theoretical framework of the Three Worlds of Mathematics.

> O The concept of mathematical research, as a teaching-learning activity, helps to bring the spirit of genuine mathematical activity into the classroom, thus constituting a powerful educational metaphor. The student is called upon to act like a mathematician, not only in formulating questions and conjectures and carrying out proofs and refutations, but also in presenting results and in discussing and arguing with his colleagues and the teacher (PONTE; BROCARDO; OLIVEIRA , 2009, p. 23).

For Ponte, Brocardo and Oliveira (2009), in the dynamics of Mathematical Research the participation of students is of fundamental importance, it requires involvement of the participants in the development of the learning process with a view to building the mathematical knowledge under study. And, in addition, Ferruzi and Costa (2018) point out that the process of working with Mathematical Research in practice leads students to interact with mathematical concepts present in the conjunction of scenarios that encompass the proposed activities, with other mathematical concepts that emerge from the discussions between students and teacher.

In the meantime, according to Visintainer (2019), the theoretical framework of the Three Worlds of Mathematics meets the formulation and interpretation of conjectures and conjunctions of the mathematical content to be investigated, articulating it in three stages of cognitive development, through the transition from mathematical thinking of each traveler through the unique ways of acting on the mathematical objects under study in each world.

> Fundamentally, the theory of the Three Worlds of Mathematics presents us with how a human being, in his development, makes mathematical connections and how knowledge structures that grow over time in size and sophistication are developed (VISINTAINER, 2019, p. 31).

For Tall (2020), the Embodied World aggregates human actions into mathematical objects (initially real that reflect in the constitution of mental objects) arising from perceptions, such as, for example, from the observation, manipulation and measurement of any cube (real object) the formulas (mental objects) are constructed for the calculation of its volume and its total surface.

According to Tall (2020), the Symbolic World is the place of human actions in mathematical objects such as, for example, numerical or algebraic expressions that trigger human actions through increasingly sophisticated symbolic manipulations, which promote the improvement of mathematical thinking with a view to developing a Formal Axiomatic System (axioms, theorems and mathematical proof), which, through Mathematical Logic and Set-Theory, constitutes the scope of the Formal World (TALL, 2020).

Ferruzi and Costa (2018) point out that mathematical research as a pedagogical practice acts to expand or improve the multiple capacities of students, “such as creativity, interpretation, reflection, argumentation, systematization and autonomy” (FERRUZI; COSTA, 2018, p. 298). With regard to the training process of Mathematics teachers, “opportunities to come and go between the Three Worlds of Mathematics should be encouraged by teacher trainers, in any content worked on in the disciplines” (BISOGNIN; BISOGNIN; LEIVAS, 2016, p. 374). It is agreed that,

> […] professors of Mathematics Degree courses, as well as researchers interested in investigating teaching and learning in such courses, deepen their studies on Tall's ideas, especially on the Three Worlds of Mathematics (SOARES; CURY, 2017, p. 81).

Bueno and Viali (2019) highlight that both the research and the planning of Mathematics teaching practices, having as an approach perspective, the theoretical framework of the Three Worlds of Mathematics makes it possible: to deepen in the mathematical object under study, to encompass resolution ideas of the problem that emerge from the discussions, to show connections between the areas of mathematical knowledge (Geometry, Arithmetic, Algebra, among others), in addition to presenting conjunctions between the different forms of representations of the mathematical object during the process of Mathematical Research, that is, it is o “Mathematics teaching with a view to allowing students to learn mathematics by doing mathematics” (FERRUZI; COSTA, 2018, p. 298).

In this context, below, the comprehensive, complex, versatile and dynamic scenario of the Three Worlds of Mathematics is discussed, culminating in the synthesis of this theoretical framework developed by Polegatti, Savioli and Leivas (2022), based on Polegatti (2020), based on reflections on the theoretical contexts derived from Tall (2007; 2013; 2020), in line with the mathematical research process within the scope of the initial training of Mathematics teachers.

## MATHEMATICAL RESEARCH AMONG THE THREE WORLDS OF MATHEMATICS IN THE INITIAL TRAINING OF MATHEMATICS TEACHERS

The complexity, dynamics and diversity of mathematical knowledge are present in the theoretical framework of the Three Worlds of Mathematics "being something broader than the addition of new knowledge to the subject's previous base" (FLORES; LIMA; MÜLLER, 2020, p. 1343). According to Polegatti and Savioli (2021b), discussions in the contexts of teaching and learning processes in Mathematics, which involve the theoretical framework of the Three Worlds of Mathematics, provide a dialogue between teaching methodologies and learning theories discussed in the context of Mathematics Education.

According to Ponte, Brocardo and Oliveira (2009), the student must become actively involved in the learning process that has objectives outlined in the teacher's planning, and that leads each student to use their cognitive apparatus, as well as to consider their affective resources for the construction of knowledge. “By requiring the student's participation in forming the questions to be studied, this activity tends to favor their involvement in learning” (PONTE; BROCARDO; OLIVEIRA, 2009, p. 23). The epistemological journeys of teachers and students, in the midst of the three mathematical worlds, promote cognitive development, encourage the active participation of the student and lead to the improvement of the mathematical thinking of each traveler (POLEGATTI; SAVIOLI, 2021a).

For Marcatto (2021), the process of learning Mathematics involves the need for students to develop mathematical reasoning with dexterity and naturalness in the face of solving activities discussed with the teacher. To this end, “teachers need to understand the nature of mathematical reasoning and be able to use teaching strategies that promote it in students” (MARCATTO, 2021, p. 2). Ponte, Mata-Pereira and Henriques (2012) emphasize that the establishment of connections inherent to mathematical knowledge by students and academics of the Degree in Mathematics is essential in the process of “thinking mathematically and, consequently, reasoning mathematically” (PONTE; MATA-PEREIRA; HENRIQUES, 2012, p. 362).

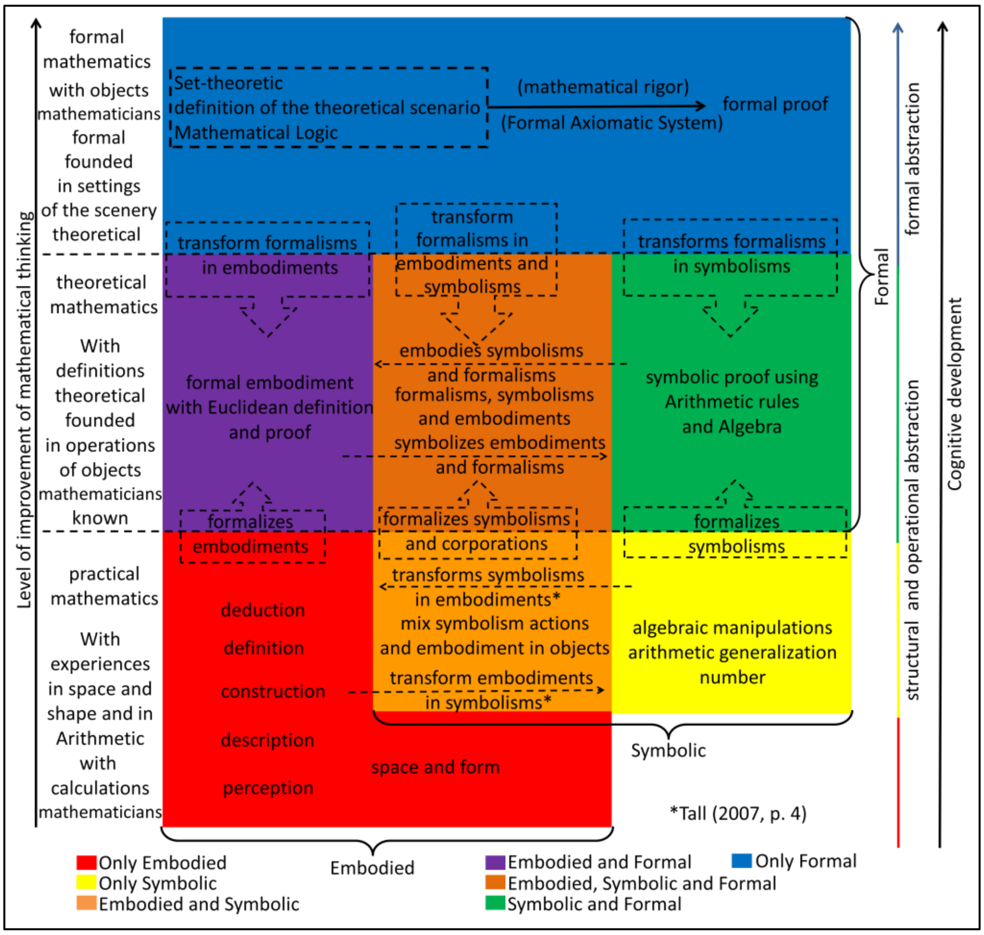
Flôres (2020) points out that studies related to the theoretical framework of the Three Worlds of Mathematics with academics of the Degree in Mathematics, provide the understanding that there are different ways of thinking mathematically (Embodied, Symbolic and Formal) before one or more activities and, emphasizes that through this theoretical framework, the teacher pays attention to “the way students think and provides different approaches to the same concept, taking into account the individual journey of each student in their evolution of mathematical thinking” (FLÔRES, 2020, p. 161).

According to Tall (2013), the Embodied World is based on human perceptions and actions on real mathematical objects (geometric solids, for example) that develop into mental representations (expressions and equations, for example) that are described in mathematical language increasingly improved, transforming themselves into mental entities present in the context of mathematical ideas.

Tall (2013) points out that the mathematical components, characteristics of the Symbolic World, emerge through numerical and/or algebraic manipulations in flexible mathematical objects (expressions, equations, functions, among others) with a view to improving mathematical language and thinking or reasoning mathematically. In the Formal World, the development of mathematical knowledge is promoted through axiomatic language based on axioms, theorems, Set-theoretic and Mathematical Logic with a view to unfolding mathematical proof (TALL, 2013).

Still according to Tall (2013), within the theoretical framework of the Three Worlds of Mathematics, the dynamics of movement of mathematical thinking is constituted in three levels of sophistication of human actions in mathematical objects (practical mathematics, theoretical mathematics and formal mathematics) by three degrees of cognitive development (structural abstraction, operational abstraction and formal abstraction). For Tall (2007; 2013; 2020), according to Figure 1, in the conjunctions of these levels and degrees, the mathematical thinking of each student and teacher is improved in the midst of the development of mathematical activities with a view to passing through the three mathematical worlds and their zones of confluences when formalizing embodiments, formalizing symbolisms and embodiments, formalizing symbolisms, embodying formalisms, transforming formalisms into embodiments and symbolisms and transforming symbolisms into formalisms.

Figure 1 – An overview of the Three Worlds of Mathematics




Source: Adapted from Polegatti (2020) and Polegatti, Savioli and Leivas (2022)

In the scenario of Figure 1, according to Polegatti, Savioli and Leivas (2022), human actions of practical mathematics (observation, manipulation, classification, construction, description, generalization, among others) in real mathematical objects (geometric solids, for example) or built in the cognitive of each student (triangles, rectangles, circles, numerical or algebraic expressions, equations, among others), instigate the development of structural abstraction with or without indications of operational abstraction, both in the context restricted to the Embodied World (red region of Figure 1), and in the specific interior of the Symbolic World (yellow region in Figure 1) and, when passing through the confluence zone of these two worlds (light orange region in Figure 1) with the mixture of characteristics of both.

According to Polegatti, Savioli and Leivas (2022), human actions in mathematical objects, at the level of theoretical mathematics, are articulated in the midst of confluence zones between the three mathematical worlds with intensity of symbolic manipulations that, in turn, promote the development of operational abstraction with evidence of structural abstraction and with a view to formal abstraction. According to this scenario,

> […] as students internalize the need to justify their statements and as their mathematical tools become more sophisticated, it becomes easier to carry out small mathematical tests (PONTE; BROCARDO; OLIVEIRA, 2009, p. 38).

In this sense, before the actions of theoretical mathematics, the embodiments are being formalized (purple region of Figure 1) through the study of Euclidean definitions and consequent development of geometric proof. In turn, the symbolisms are being formalized (green region in Figure 1) through increasingly sophisticated algebraic manipulations with the purpose of constructing an algebraic proof. In the triple zone of confluence between the three worlds (dark orange region of Figure 1), there is the process of formalizing the consonances between embodiments and symbolisms. With regard to the Formal World, it is agreed that:

> Introducing the idea of mathematical proof can be done gradually, restricting yourself, in an initial phase and with younger students, to looking for an acceptable justification, which is based on plausible reasoning and on the knowledge that students have (PONTE; BROCARDO; OLIVEIRA, 2009, p. 38).

Thus, the Formal World (blue region of Figure 1) corresponds to the level of human actions on formal mathematical objects (formal mathematics). Based on Tall (2013), in this world, geometric and/or algebraic proofs are improved for mathematical proof through the construction of a Formal Axiomatic System, which uses axioms and theorems in conjunction with Set Theory and Mathematical Logic, with views to the development of formal abstraction in each traveler. That is, the individual thinks or reasons mathematically when performing manipulations with formal mathematical objects that operate with an axiomatic formal language.

Next, the resolution is described and an overview of the discussions between the teacher trainer and the three participating academics involving the teaching and learning process of solving second degree equations is presented, in view of the theoretical framework of the Three Worlds of Mathematics, based on a problem chosen by the participants.

## MATHEMATICAL RESEARCH AND TEACHING THE FORMULA FOR RESOLVING SECOND DEGREE EQUATIONS BETWEEN THE THREE WORLDS OF MATHEMATICS

Polegatti and Savioli (2021b) emphasize that the Mathematics teacher, already in the process of elaborating his teaching plan based on the theoretical framework of the Three Worlds of Mathematics, completes his cognitive journey through the three distinct, organized and connected modes of development of the mathematical knowledge (Embodied, Symbolic and Formal). Remembering that "[...] not all individuals go through the Three Worlds of Mathematics in the same way, because there are different influences, from their environment, their schooling and the difficulties they encountered in their learning" (BISOGNIN; BISOGNIN; LEIVAS, 2016, p. 366).

With regard to the dynamics of Mathematical Research in teaching and learning processes of Mathematics,

> A mathematical research activity usually takes place in three phases (in a class or set of classes): (i) introduction of the task, in which the teacher makes a proposal to the class, either orally or in writing. (ii) carrying out the investigation, individually, in pairs, in small groups or with the whole class, and (iii) discussion of the results, in which students report their work to colleagues (PONTE; BROCARDO; OLIVEIRA, 2009, p. 25).

In this sense, in compliance with the first phase of the Mathematical Research dynamics, the professor trainer responsible for the subject of Practices of Elementary School, proposes to the academics the study of the formula for solving second degree equations. In this curricular component, the aim is to discuss with future Mathematics teachers the teaching processes of Mathematics content present in the curricula of the final grades of Elementary School (fifth to ninth grade), linking to them theories of Mathematics learning. It is agreed that,

> The Three Worlds of Mathematics are not hierarchical, they mix and complement each other in the process of learning Mathematics, and these experiences are fundamental for the development of mathematical thinking (MARTELOZO; SAVIOLI, 2019, p. 59).

It should be noted that "from the combination of three distinct and complementary worlds, we emphasize the importance of experiences involving mathematical concepts in different contexts" (MARTELOZO; SAVIOLI, 2019, p. 60). Contexts that both involve different areas of mathematical knowledge (Geometry, Arithmetic, Algebra, among others) and those related to the problem(s) used as a source of discussions involving mathematical content(s) to be (in) studied. Mathematical Research's in the classroom involving geometric contexts contribute to the perception of "essential aspects of mathematical activity, such as the formulation and testing of conjectures and the search and demonstration of generalizations" (PONTE; BROCARDO; OLIVEIRA, 2009, p. 71). Furthermore, as in the Three Worlds of Mathematics, the dynamics of mathematical research can contribute,

> [...] to materialize the relationship between situations of reality and mathematical situations, to develop abilities, such as spatial visualization and the use of different forms of representation, to show mathematical connections and to illustrate interesting aspects of the history and evolution of Mathematics (PONTE; BROCARDO; OLIVEIRA, 2009, p. 71).

In this sense, there is the following problem for this investigation: "Pedro used 500 meters of wire to go around, once, a rectangular pasture of 1 hectare. What are the dimensions of this pasture?" (BARROSO, 2006, p. 61). It is argued with academics that in solving this problem different mathematical contexts can be involved (geometric, arithmetic, algebraic and formal), which gain notoriety and connections in the dynamics of mathematical research within the theoretical framework of the Three Worlds of Mathematics.

Na In the second phase of mathematical research, the joint discussion of academics is encouraged within the scope of the confluence zone between the Three Worlds of Mathematics (dark orange region), which, when affected by their actions in theoretical mathematics, leads to the development of operational abstraction with hints of structural abstraction. In this context, we seek to build a geometric representation of the problem by drawing a rectangular region, denoting the variables $x$ and $y$ for its length and width, respectively. And, after some symbolic manipulations, the relations $y = 250 - x$ are reached, considering the perimeter of the pasture and, $x.y = 10{,}000$ for the calculation of its area. It is noteworthy that these representations are present in the resolution of the textbook used.

> In fact, the challenge launched by the generalization of a numerical pattern and the understanding of what this generalization translates are aspects that are often involved in numerical investigations and that support the development of algebraic reasoning (PONTE; BROCARDO; OLIVEIRA, 2009, p. 69).

From this perspective, the discussions continue within the confluence zone between the Embodied and Symbolic worlds (light orange region in Figure 1). Through practical mathematics actions, students are led to calculate some values for these dimensions (here, five values are suggested). Then, after the numerical calculations that promote the development of structural abstraction with indications of operational abstraction, the result is that, given what is informed in the problem, the length of the pasture measures $50\,m$ and its width $200\,m$. Then, the teacher intervenes to question whether these values are the only ones that satisfy the resolution of the problem.

Ponte, Brocardo and Oliveira (2009) point out that the process of mathematical creation is dynamic and involves back and forth movements between areas of mathematical knowledge, in contrast "with the usual image of this science, as a body of knowledge organized in a logical and deductive, like a solid building, a paradigm of rigor and absolute certainty" (PONTE; BROCARDO; OLIVEIRA, 2009, p. 15). In this sense, the discussions return to the scope of the confluence zone between the three mathematical worlds, through joint actions of theoretical mathematics, which enable the development of operational abstraction with evidence of structural abstraction and with a view to formal abstraction. From this dynamic emerges the quadratic equation $-x^2 + 250x - 10{,}000 = 0$, arising from the symbolic manipulations between the perimeter and area relations of the referred pasture.

According to Lima, Healy and Koch (2017), algebraic manipulations for solving quadratic equations mainly involve characteristics of the Symbolic World. So, in view of the illustrations in Figure 1, within the Symbolic World, through practical mathematics actions, the quadratic equation is solved, promoting the development of structural abstraction with indications of operational abstraction, whose roots are 50 and 100 going to meeting what was previously calculated. After this observation, the professor invites the academics to analyze what has been done so far. It is concluded that the way the problem was solved matches exactly what is suggested in the textbook used. However, in the context of Mathematical Research, it should be noted that

> When we work on a problem, our goal is naturally to solve it. However, in addition to solving the proposed problem, we can make other discoveries that, in some cases, prove to be as or more important than the solution of the original problem (PONTE; BROCARDO; OLIVEIRA, 2009, p. 17).

Thus, within the Embodied World (red region of Figure 1), through practical mathematics actions, rectangular representations of the pasture calculated by the academics (the five suggested), without the inclusion of their measurements, which help in the development of structural abstraction. After all, as Tall (2020) states, visual representation helps in the process of learning mathematical knowledge.

Continuing with the discussions, we seek to formalize, geometrically, the calculation of the area of a rectangular region. Thus, with the discussions within the purple region of confluence between the Embodied and Formal worlds (according to Figure 1), it is proposed to square a given rectangle in such a way that the sum of the areas of its squares can be determined by the product between the amount of squares in the horizontal position and the total squares in the vertical position. This theoretical mathematics action illustrates the calculation of the area of a rectangular region, promotes the development of operational abstraction with evidence of structural abstraction and with a view to formal abstraction.

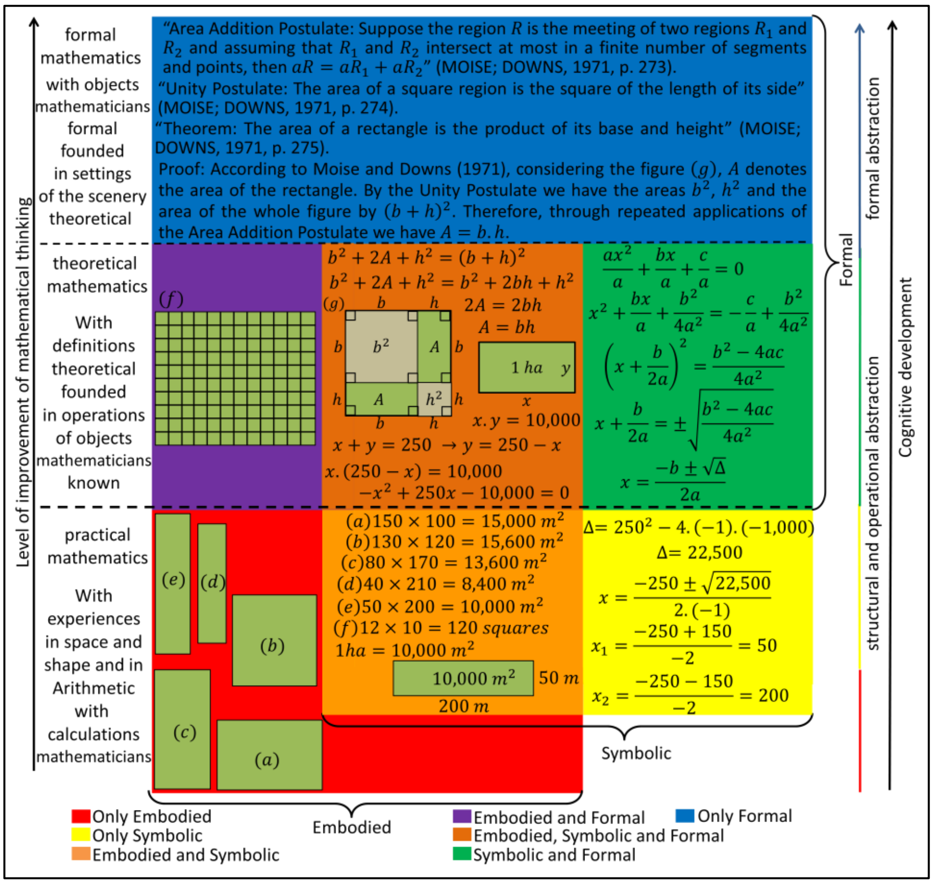
In the dynamics of Mathematical Research, for Ponte, Brocardo and Oliveira (2009), it is necessary to pay attention to the refinement processes of the resolutions already formulated, to elaborate other conjectures, as well as to carry out demonstrations in the midst of formal manipulations. In this perspective, through actions of theoretical mathematics, the algebraic demonstration of the formula for solving quadratic equations is developed within the confluence zone between the Symbolic and Formal worlds (green region in Figure 1). These symbolic manipulations promote the unfolding of operational abstraction, with evidence of structural abstraction and with a view to formal abstraction.

Inside the Formal World, based on Moise and Downs (1971), research is carried out on formal mathematical objects (postulates and theorem) that support the calculation of the area of a rectangular region. In this mathematical research, through formal mathematical actions with a view to the development of formal abstraction, there is the immersion of the Postulate of Addition of Areas, the Postulate of Unit and the Theorem that formally bases the geometric procedure of calculating the area of a region rectangular.

In turn, the demonstration of the theorem is built within the confluence zone between the three mathematical worlds through theoretical mathematics actions by the participants, with the development of operational abstraction (algebraic manipulations), amidst evidence of abstraction structural by involving the representation of the design of the rectangular region, as a sum of square and rectangular areas (Postulate of Unity in line with the Postulate of Addition of

Areas) and, with a view to formal abstraction in the midst of the construction process of the mathematical proof of mentioned theorem. Figure 2 illustrates the representations arising from the discussions within the Three Worlds of Mathematics.

Figure 2 – Discussions from the perspective of the Three Worlds of Mathematics




Source: Adapted from Polegatti (2020)

According to Tall (2020), the distinctions between practical, theoretical and formal mathematics are useful in the teaching and learning processes of Mathematics. For the author, formal mathematics, for example, is conceptually more powerful than practical and theoretical mathematics because it can be applied in any mathematical context in which the axioms and theorems are valid, including any future educational scenario still unknown that satisfies the axioms and their definitions. The human actions of formal mathematics,

> [...] it applies to future evolution of ideas in axiomatic formal mathematics, always with the possibility that we will develop even more sophisticated ways of thinking in the future that we have not yet considered. [...] the future is already here in the sense that it is possible to prove formal structure theorems that take us on to more sophisticated forms of embodiment and symbolism. This can still be encompassed in the current framework with three forms of mathematics (embodiment, symbolism, formalism) in three levels of sophistication (practical, theoretical, axiomatic formal) (TALL, 2020, p. 3-4).



In this sense, according to Tall (2020), formal mathematics and formal abstraction represent the apex of mathematical knowledge, but not its end, since human actions on formal objects (postulates, axioms and theorems) have repercussions on the actions of individuals on objects mathematicians under study through theoretical mathematics and or practical mathematics. In this scenario, it is understood that teaching and learning processes of Mathematics, in perspective in the theoretical framework of the Three Worlds of Mathematics, favor the development of Mathematics Research practices in the classroom, whether numerical, geometric, algebraic or formal, as well as their consonances.

As Polegatti and Savioli (2021a) point out, the methodology of Mathematical Research in line with the theoretical framework of the Three Worlds of Mathematics, promotes interaction between the mathematical objects under study, so that each traveler's mathematical thinking or reasoning develops not in a linear way, but dynamic in its complexity and flexibility, spontaneously or provoked and with articulated movements between the worlds and their zones of confluence.

## FINAL CONSIDERATIONS

In view of what was discussed, it is concluded that the dynamics of the Mathematical Research methodology in practice, in line with the theoretical framework of the Three Worlds of Mathematics, favors the planning and teaching and learning processes mediated by debates between the teacher and his students , in the midst of practical, theoretical and formal mathematics actions in the mathematical object under study with a view to the development of mathematical abstractions (structural, operational and formal).

It is worth mentioning the research by Lima, Healy and Koch (2017) which, through the investigation of learning from the perspective of the theoretical framework of the Three Worlds of Mathematics, highlights the difficulties of the participating students in the ninth year of Elementary School, in the process of transition from solving linear equations to quadratic equations. “Such difficulties involve both embodied characteristics, in attempts to evaluate equations or use graphs, and symbolic ones, when seeking algebraic manipulation to solve equations” (LIMA; HEALY; KOCH, 2017, p.766).

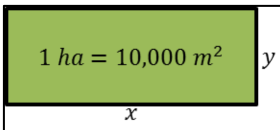
Lima (2019) points out that discussions involving characteristics of the Formal World do not axiomatically apply in the scope of Elementary Education (in this case, the debate with ninth grade students who study the resolution of high school equations). “However, in any mathematical activity there are characteristics of the formal world, as a student does not build mathematical concepts without understanding their formal characteristics” (LIMA, 2019, p. 8). In addition, the author points out that working in the classroom with mathematical activities that promote the transition of each student's mathematical thinking or reasoning between the Three Mathematical Worlds is essential for the construction of mathematical knowledge.

Still regarding the scope of the Formal World, in the case of research with undergraduate students in Mathematics, according to Flôres (2020), the study of the formal representations that underlie one or more Mathematical objects are fundamental for the construction of arguments by Mathematics teachers in its validation procedures for its students' mathematical solutions, as well as to resolve doubts and answer questions raised by its students.

After all the discussions, it was verified that the problem used by Barroso (2006) brings inconsistency in the addressed context. The solution that the sides of the rectangular pasture are $50\ m$ and $200\ m$ before $10{,}000\ m^2$ of area is valid in the mathematical context. But, are there pastures fenced with just one wire loop? When investigating this question, one comes to the conclusion that, usually, a pasture is surrounded by a fence composed of five turns of smooth wire.

In this sense, the measure of $500\ m$ of wire informed in the problem must be converted to $100\ m$ per turn, when considering the context of a fence composed of five turns of wire. Given this reality, would it be possible to fence off the rectangular pasture, whose area measures $1$ hectare ($10{,}000\ m^2$), with the amount of screen provided in the problem? The mathematical representations of the new calculations are presented in the scenario of Figure 3.

Figure 3 – Resolution of the problem with the fence composed of five turns of wire




$$2x + 2y = 100 \qquad x.y = 10{,}000$$
$$x + y = 50 \qquad x.(50 - x) = 10{,}000$$
$$y = 50 - x \qquad -x^2 + 50x - 10{,}000 = 0$$
$$\Delta = b^2 - 4ac = 50^2 - 4.(-1).(-10{,}000) = -37{,}500$$
$$\Delta < 0 \rightarrow x \notin \mathbb{R}$$

Source: The authors (2022).

Therefore, the $500\ m$ of flat wire are insufficient to surround a pasture with an area of 1 ha with five turns. So, another question can be raised: how many meters of wire are needed to build a fence composed of five wires to surround a rectangular pasture with an area of $10{,}000\ m^2$ and with the same measures presented in the resolution of the problem in the textbook? After some arguments, it is concluded that $2{,}500\ m$ of wire are needed to have the same dimensions, $50\ m$ and $200\ m$, on the sides of the rectangular pasture.

It should be noted that in the 2018 version, the wording of the problem was changed “With $500\ m$ of canvas, Mariana contoured a rectangular plot of $10{,}000\ m^2$. What are the land dimensions? (GAY; SILVA, 2018, p. 184). The wire was replaced by the screen, implying the need to take only one turn to enclose the land, and the agrarian measurement of the hectare was removed from the question. Finally, it is important to look at the whole picture of a problem, its information, its possible resolutions, interpretations and the context involved.

In this sense, it should be noted that Mathematical Research activities in line with the Three Worlds of Mathematics favor looking at the general picture, of teachers and students and, with attention to the details of contexts and representations (embodied, symbolic and formal) of mathematical objects under investigation in the midst of the teaching and learning processes of Mathematics.

# INVESTIGAÇÃO MATEMÁTICA EM MEIO AO QUADRO TEÓRICO DOS TRÊS MUNDOS DA MATEMÁTICA NA FORMAÇÃO DO PROFESSOR DE MATEMÁTICA

**RESUMO**

Neste trabalho, tem-se o objetivo de discutir a consonância entre o processo de Investigação Matemática em sala de aula e o quadro teórico dos Três Mundos da Matemática (Corporificado, Simbólico e Formal) em meio à formação inicial de professores de Matemática. Trata-se de uma pesquisa de natureza bibliográfica, na qual, por meio de análises textuais (livros, teses, dissertações, artigos, entre outros) promove-se o encontro teórico entre dois ou mais campos de estudo. Por conseguinte, debate-se, hipoteticamente, a resolução de equações do segundo grau, por meio de sua fórmula específica, tendo como possíveis interlocutores, acadêmicos de um curso de Licenciatura em Matemática, no âmbito da disciplina de Práticas do Ensino Fundamental. Então, desenvolvem-se diálogos na perspectiva metodológica de Investigação Matemática, com vistas a envolver as três vertentes matemáticas distintas de se agir em objetos matemáticos (prática, teórica e formal) e, entrelaçadas aos três níveis de abstração do conhecimento matemático (estrutural, operacional e formal). As análises das discussões, com base no referencial teórico adotado, permite-se concluir que a inserção da prática de Investigação Matemática em consonância com o quadro teórico dos Três Mundos da Matemática possibilita ao professor pesquisar, planejar, agir matematicamente e, analisar conjeturas de resoluções de um ou mais problemas, bem como fomentar e validar as discussões e resoluções apresentadas por seus estudantes em meio ao conjunto de áreas do conhecimento matemático (Aritmética, Geometria, Álgebra, Geometria Analítica, entre outras), de forma a ressaltar particularidades e ou destacar as similaridades e conjunções entre elas.

**PALAVRAS-CHAVE:** Educação Matemática. Investigação Matemática. Três Mundos da Matemática.

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