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Investigating topological properties with elementary school students

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ABSTRACT

This article is the result of a research originated in a Study and Research Group in Geometry, which has been dedicated to exploring alternative methodologies that can contribute to this area, including through the development of appropriate teaching resources, whose application should happen in locus, in order to validate it and suggest proposals. This work aimes to analyze how elementary school students understand facts of a non-Euclidean geometry - topological geometry - exploring concrete didactic material resources. Six activities were applied involving material resources such as: coloring maps with minimal amount of colors; modeling clay, in order to conclude the mass conservation; strings, for continuous transformations of closed lines; Möebius Band, to analyze the continuity property; balloons, to illustrate continuous deformations, among others. It was possible to investigate the students' understanding of the theme at this level of education. It was concluded from the research that it is possible to introduce basic topological notions that tend to contribute to the geometric formation of the students, which includes the transition to high school geometry, for example, when studying the Euler's relation to polyhedral.

KEYWORDS: Topology. Elementary School. Möebius band.

INTRODUCTION

The best known geometry addressed in schools is that of Euclid, which emerged around the 3rd century BC It uses axioms and deductive thinking in a rigorous way, but it was around the 20th century that new geometries appeared, which did not follow the Euclidean axiomatics. These new geometries became known as non-Euclidean geometries, among which are: topological, hyperbolic, spherical, projective and fractal.

In this work, in an informal way, it was intended to explore elementary notions about topological geometry with an elementary school class. The RBECT scope predicts the dissemination of research that aims the teaching-learning process. Thus, the research contained in this article was carried out, in which pedagogical strategies for teaching one of the non-Euclidean geometries are explored with a 9th grade class of elementary school. Motivated by the work carried out in a study and research group in Geometry, led by the second author and in which the first author has participated since its creation, we sought to apply the knowledge and alternative methodologies to a 9th grade class at this level of schooling.

Through the investigation and understanding of some properties of this type of geometry, we sought to understand, intuitively, what topology is, without, however, deepening scientific knowledge about it. Thus, this research is justified, which aimed to analyze how elementary school students understand facts of a non-Euclidean geometry - the topological, exploring concrete didactic material resources.

Topology is an area of mathematics known informally as rubber geometry. In it, figures located in space that have topological properties are studied, that is, those that do not vary when they are continuously deformed without breaking. For example, when using an eraser written on paper, it is common to deform it (most common are those in the shape of a rectangular parallelepiped) and to observe the variation of its shape until it tends to break (its faces are no longer flat). When the rubber breaks, points that were close together may be far apart, so that the neighborhood property is lost, losing its topological feature. For Pinto (2004, p. 02) "a topological transformation is a transformation of one figure into another in such a way that any two points that are together in the original figure remain together in the transformed figure".

One type of transformation of this nature that does not alter topological features is to stretch or lengthen, for example, a balloon of the kind used for birthdays. Two dots are marked with a pen and, when filling it, it is possible to see, regardless of the quantitative aspects such as length, distance, area or volume, that the qualitative ones, such as the concepts of interior/exterior, inside/outside, the order of the stitches and the amount of material remain unchanged. Thus, in topology, some transformations are allowed, such as stretching/stretching, shrinking, bending and twisting. It is a very broad area and can be divided in a basic way as general topology, algebraic topology and geometric topology.

Topological geometry studies surfaces with a dimension less than or equal to 4, as well as their applications to bundles (lines that vary from point to point in space), including node theory. The 1- and 2-dimensional spaces have special names: a 1-dimensional variety is called a curve; a 2-dimensional variety is called a surface.

A curious aspect of the topology is that there is no difference in terms of size between a cup and a log (like a tire inner tube), as they both have the same number of holes (in the cup corresponds to the handle). These two objects can change into each other continuously, without disruption. Therefore, they are considered topologically equivalent.

According to Druk (2011), topological spaces are distinct. For example, if you take a ball (spherical surface) or an air chamber (torus) and draw a circumference on each of them, you will notice that in the ball, every circumference over it limits a disk contained in that surface, ie , an upper part on the ball and a lower part, so to speak. If it were massive and cut by such a circumference, a disk would appear. For the air chamber, not every circumference will be the contour of a disk contained in this surface, as this, when cut, will not maintain such shape.

Möebius, Listing and Gauss were the forerunners of the topology that emerged with the development of geometry in the 19th century. Druk (2011) attributes the beginning of this part of the geometry to the work of Leonard Euler on the Konigsberg bridges, around 1973. In the path traveled along the river, from the exit point to the return, passing through all existing bridges along the route, the distance was not relevant to the solution, but the condition of passing each bridge only once was. This problem triggered Euler's formula, which demonstrates that, regardless of the size of a polyhedron, it is possible to have the number of vertices minus the number of edges plus the number of faces equal to two. By itself, this evidences a new geometry, different from the Euclidean one.

According to Silva and Leivas (2013), there are three classic topology problems: (1) Konigsberg bridges; (2) the four-color theorem; (3) the Möebius Band (or Band or strip, usual expressions). These are connected because, according to Laranjeiras,

> the German mathematician and astronomer August Ferdinand Möebius (1790-1868), [...] studied this object in 1858 motivated by a competition promoted by the Paris Academy of Sciences which, at the time, was stimulating the study of the geometric theory of polyhedra, geometric solids whose surfaces are composed of a finite number of faces. The object ended up being popularly known as the “Möebius Ribbon” (LARANJEIRAS, 2010, p.01).

To build the Band, you can take a 5cm x 30cm paper tape, for example, and give it a half turn at one end, in order to sew it to the other (5cm ends). From this, a curious and intriguing fact can be observed: the Strip has only one side and one edge! In other words, it is not possible to distinguish which is the inner and the outer side of this one, which means to be a surface with only one side, that is, a non-orientable surface. Studies and searches for mathematical solutions to understand the shape and properties of the tape were much investigated in the 21st century.

The mathematician Johann Benedict Listing (1808-1882), who was an architect, also studied this form that today can be found in works of art, jewelry, music, architecture, psychoanalysis, cell biology, liquid crystals, robotics, materials engineering, etc. . A very practical example where the Möebius Strip can be found is in the recycling symbol, which is generated from the notion of triangulation of the strip.

The Möebius Ribbon gained prominence in the world of Psychoanalysis with the French Jacques-Marie Émile Lacan (1901-1981), who used the Ribbon as a model for the representation of the human psyche. In Leivas, Bettin, Franke (2019), there is an investigation with this approach.

The Phoenix International Media Center, located in Chaoyang Park, Beijing[1], is a beautiful example of architecture that uses the concept of "Möebius' Band" in its sustainable construction, and started from the idea of integrating people from both sides of the building. (Figure 1).

Figure 1 – Example of figure

Source: Primary data (2020).

Other applications can be seen in an article in which authors of this study present an investigation involving possibilities of use and application of the Möebius Band in different areas Leivas, Bettin, Franke (2019).

METHODOLOGY

In order to outline this work, we sought support in Bauer, Gaskell and Allum (2015), who distinguish four dimensions for a social investigation, such as this one, which sought to analyze how elementary school students understand facts of a geometry non-Euclidean - the topological -, exploring concrete didactic material resources. Thus, the authors describe, in the dimension, the research design as: "sampling survey, participant observation, case studies and experiments and quasi experiments" (p. 19). The present research can be understood as a combination of these elements, since the researchers actively participated in the study, guiding individuals in carrying out the tasks. Furthermore, the investigation took into account a specific group of students who experienced concrete activities involving topological elements.

With regard to the second dimension indicated by the authors (BAUER; GASKELL; ALLUM, 2015), there is the process of data collection, which in this case corresponds to the search for documents originated by the participants and researchers for further analysis. The authors (Idem, 2015) put as a third dimension: "there are the analytical treatments of data, such as content analysis, rhetorical analysis, discourse analysis and statistical analysis" (p. 19), and the research in question will stick to the analytical treatment of the collected data. The authors conclude by indicating the interests of knowledge.

The activities to be described, whose data were collected and later analyzed, were carried out in the first half of 2019, with a group of 13 students from the 9th grade of elementary school from a state school in a city in the central region of Rio Grande do Sul – RS. The head teacher of the class and author of the article was motivated to apply the knowledge acquired in the research group in order to incorporate it into the class whose characteristic involves curiosity and motivation to acquire new knowledge as they prepare to enter a school that has a higher level of demand than others in its selection process.

The school's teaching plan does not include such content as it is not mandatory according to the state's curriculum. However, it was perceived the possibility of exploring constructions with concrete materials involving topological properties compared to Euclidean ones.

Activity 1

Statement - Theorem of the four colors: for this activity you will need a map of Brazil to color. Paint this map with as few colors as possible, as two neighboring regions should not remain the same color.

The purpose of this activity is to note that the minimum amount of colors that satisfy the given condition is four. This activity allows us to analyze the first elementary topological relationship, according to Piaget and Inhelder (1993), which is the neighborhood, because "[...] corresponds to the simplest condition of the entire perceptual structure, that is, the 'proximity' of the elements perceived in the same field" (p. 21). For the authors, this problem gave rise to the theory of graphs, which led to that of nodes, being important for the establishment of Euler's theorem.

NOTE: The activity was taken from the article "the construction of space according to Jean Piaget", by Lívia de Oliveira, from UNESP.

Activity 2

Statement - Topological transformation: for this activity you will need modeling clay and 30cm long string.

a) Start the activity with the string. Tie the ends together and form a square on a flat, smooth surface. Then turn it into a circle. Later, in a diamond and then in a rectangle. Finally, turn it into any closed curve. Write down your observations about the experiment.

b) Using modeling clay, make a sphere and later turn it into a disk, without breaking, tearing, adding or removing mass.

From this activity, it is intended that students understand the notions of topological transformation and the property of conservation of mass, because, before and after continuous transformation, it is maintained, that is, the two objects have the same mass, independently of size and shape. Thus, through continuous transformations, it is passed from one form to another.

Activity 3

Statement - Topological properties: for this activity you will need two square regions of different colors, obtained from cardboard. Take the regions and put a different letter in each corner. Crush one of them and tear the other into pieces. Then, unwrinkle or smooth the first one and try to piece together the second one. Look at the two regions and answer the following questions:

a) Did the two regions change or maintain their shape?

b) Did the points before and after the transformation maintain the same proximity?

For Piaget and Inhelder (1993), in this activity, the second elementary spatial relationship is explored, that is, that of separation. For the authors, “Two neighboring elements can, in effect, interpenetrate and be confused in parts: introducing between them a relationship of separation consists in dissociating them or, at least, in providing a means of distinguishing them” (p. 21). Furthermore, topological properties of transformation, proximity, shape, far/near, separate/united and continuous/discontinuous can be explored.

Activity 4

Statement - Construction of the Möebius Strip: for this activity you need three strips of paper 30cm long by 5cm wide, colored pen, scissors and glue. Take one of the strips and glue one end to the other in a ring shape. On the other two strips of paper, make a half turn at one end and seam at the other, sticking them together. Observe and answer the following questions:

a) How many sides is each ribbon?

b) If you cut the tape that has the longitudinal twist in half, what happens?

c) If you cut only a third of the other tape that has the twist, what happens?

From this activity, it is intended that the student understands the existence of geometric shapes that have only one side and one edge, which are called non-orientable surfaces. The student must also analyze the following properties: "not orientable", "only one side", "one edge", "internal and external", "increase in length without increasing the amount of material", "closed curve", “open/closed”, “notion of infinite and finite”, “difference between circle and circumference”.

Activity 5

Statement - The Balloon Experience: For this activity you will need an empty birthday balloon. Mark with the pen two points A and B on it, then fill it. Note what happened before and after filling the balloon and answer the following questions:

a) What is the topological transformation undergone by the balloon?

b) Complete Chart 1 with the words: “conserves” or “modifies”.

Chart 1 – Geometric properties obtained (Red writing represents the expected response)

Geometric Properties	Topological Geometry	Euclidean Geometry
Order of points	conserves	conserves
Interior Exterior	conserves	conserves
Lines	modifies	conserves
Angles	modifies	conserves
Distance	modifies	conserves
Material	conserves	conserves
Positions	modifies	modifies

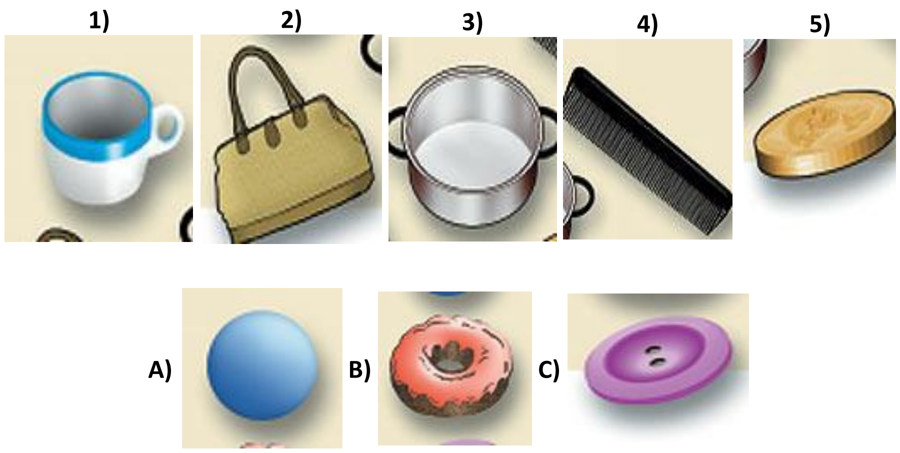
Source: Research data (2020).

Activity 6

Statement - In topology, you can deform an object (stretch or compress), but not make cuts or patches. In this situation, the geometric properties will remain constant. So a donut can be turned into a cup of coffee. The donut hole becomes the coffee cup wing hole, for example.

Following this idea, list the numbered figures with the three marked with letters (Figure 2), captured from Galileo Magazin[2].

Figure 2 – Topological configurations



Source: Captured data by authors (2020).

RESULTS AND DISCUSSIONS

In solving the first activity, which was to paint the map, the students discussed the amount of colors needed to paint it. At this stage, most responded that the minimum needed was five colors.

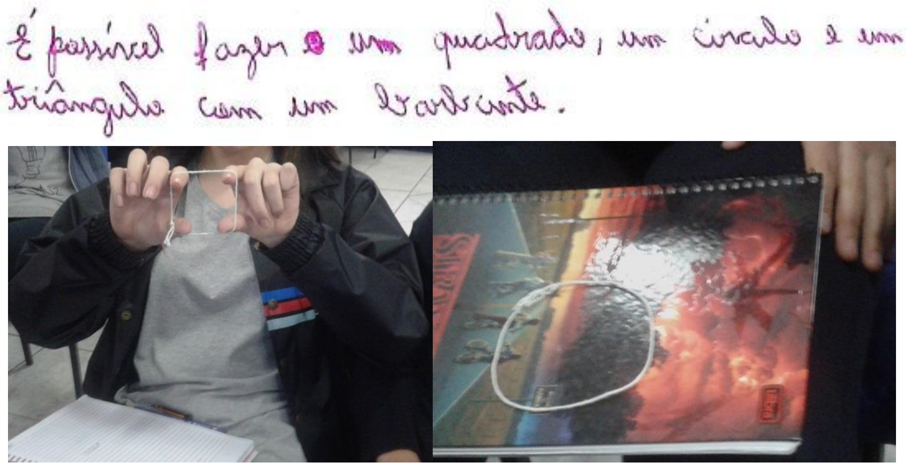
Figure 3 – Participants performing the painting map activity

Source: Authors (2020).

Some students were painting at random way, while others thought and developed strategies before painting, checking color possibilities when visualizing neighboring states on the map. It was noticed, in the first painted map of the sequence shown in Figure 3 (left), that the student tried to use a minimum amount of three colors, but when it came to painting the state of Bahia, he had to use another color.

Activity 2 illustrates what was done by student A, who tied the string and made the requested shapes, as recorded in Figure 4.

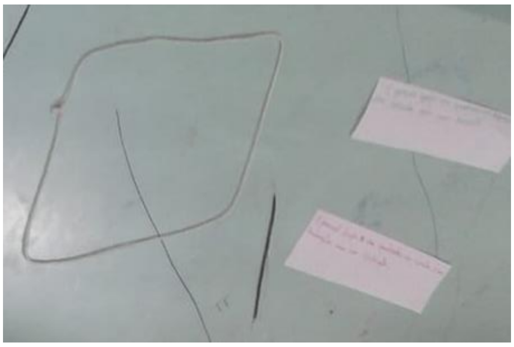
Figure 4 – Student A Record



Source: Research data (2020).

In the records shown in Figure 5, student B explained verbally and the researcher made the respective record: “No, because there is no vertex in the square”.

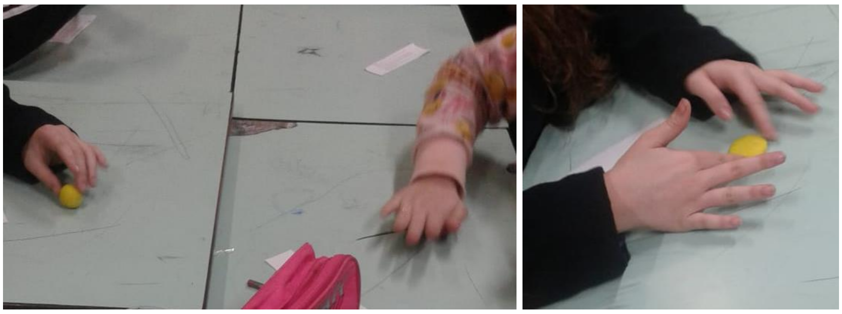
Figure 5 – Student B Records



Source: Research data (2020).

When using the modeling clay, still in activity 2, the students obtained the sphere and then turned it into a disk, without breaking, separating, adding or removing the dough. They thought the activity was easy. Figure 6 illustrates students C and D performing the task.

Figure 6 – Students making the sphere with modeling clay



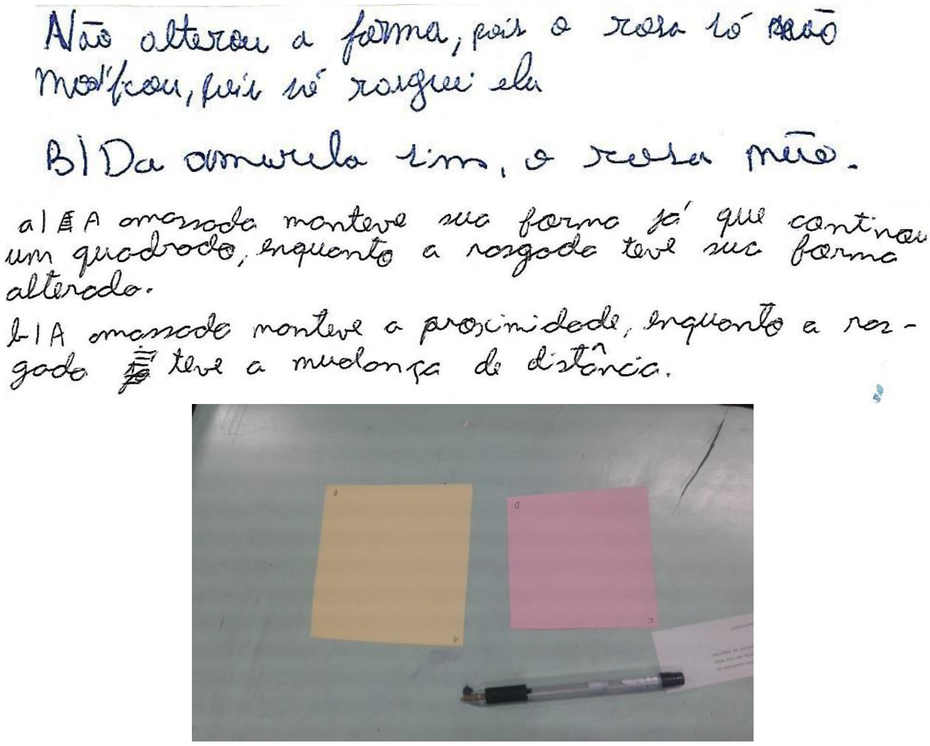
Source: Research data (2020).

Thus, through continuous transformations, the students were able to carry out the activity, concretely illustrating the respective expected topological property, that is, shape change without loss of mass. A large part of the participants said it was possible to make the transformations and found that the amount of mass before and after the transformation remained the same, regardless of the size and shape of the object.

When using the string, some students stated that it was possible to pass from one form to another. Others, however, said that this could not happen, as the vertices of the square and the triangle were not formed. The teacher agreed with the latter, stating that they were representations and not the geometric object itself, which is an abstract concept.

In activity 3, the students understood the alternative, which consisted of changing or maintaining the shape, as well as justifying whether or not there was a square after the transformation. Therefore, most said that they changed the shape, as the one that had been torn was no longer a square region, as illustrated in Figure 7.

Figure 7 – Record of activity 3a) of students E and F



Source: Survey data (2020).

In the letter b) of activity 3, it was unanimous that the points maintained the proximity in the 'square' that had been crushed, while in the one that was torn there was no maintenance of proximity, as expected in a topological transformation. It is observed that, when working with the material, the students did not distinguish a square from a square region, even because the square, which is a flat geometric figure, is in the individual's mind. Figure 8 illustrates students trying to restructure the two ways.

Figure 8 – Entire and torn rectangular region

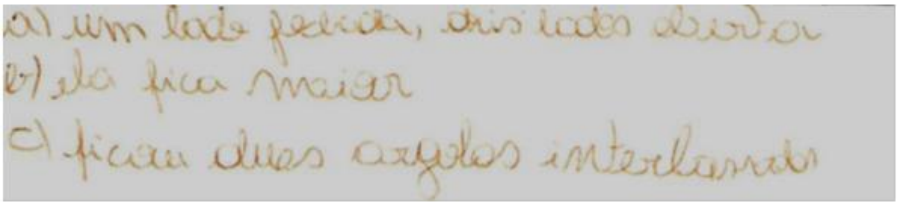
Source: Research data (2020).

After this activity, the topological properties of transformation were explored: proximity, shape, far/near, separated/united and continuous/discontinuous.

To carry out activity 4, two periods of Mathematics classes were used, during which the students built three tracks. They were asked how many sides the first one had. For most students, this question had a two-sided verbal answer, as they reported to it before the collage, according to the explanations given to the teacher after their questions.

After this process, in order to obtain the Möebius Tape, it was recommended that they paint it, which led them to conclude, with some surprise, that no, it did not have both sides, as it initially had before the collage. The result left them quite intrigued and questioning. As the teacher had commented about the Track being used for certain writings, especially in the sense of loving declarations, one student realized that there was a continuous writing, reading non-stop as she walked along the track, which left her intrigued by not noticing the initial two sides that existed before the collage. It was explained by the teacher that this happened precisely because Tape, in this situation, had only one side. In Figure 9 are the student's records for the three items of this activity.

Figure 9 – Student G's records to activity 3 items



Source: Research data (2020).

This same student meant that, when the Ribbon is closed, it has one side, and when it is open, it has two (Figure 10).

Figure 10 – Students building the Möebius Strip

Source: Research data (2020).

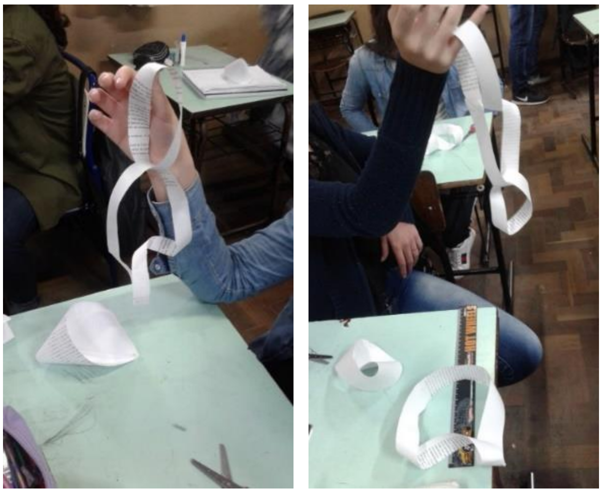
In letter b), they asked how to cut the tape in half in the longitudinal direction and asked them to check what would happen. They found that, having twisted the initial tape, when cutting it, it did not break, which, again, generated great surprise, as illustrated in Figure 11.

Figure 11 – Tape broken in the longitudinal direction

Source: Survey data (2020).

For these students, the biggest novelty came when they did what was asked for in letter c): when cutting only longitudinally, for a third of the initial width of the torsion, what appears? Figure 12 records what was done.

Figure 12 – Cutting the tape by the third of its width in the longitudinal direction

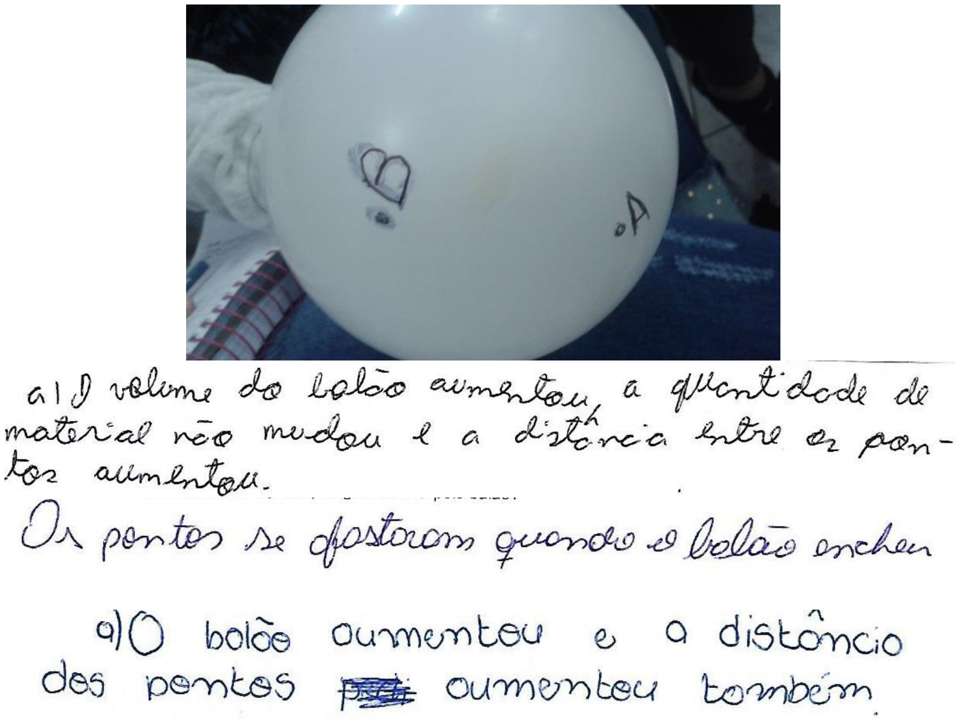


Source: Research data (2020).

This activity provided the students a curiosity, surprise and discovery moment, because before cutting the Ribbon, everyone believed that they would break it in half, making it another Möebius Strip again. When they were amazed and didn't believe in what was happening, they questioned the teacher how different forms were obtained. It was said that if the tape with the twist was cut in half, having as a reference its width in the longitudinal direction, it would be larger and not separated. On the other hand, if the cut was made by the third of the width, in the other strand with twist, there were two strands, one longer and the other shorter.

In activity 5, the students worked with a birthday balloon and noticed that, when they inflated it, the distance between the points they had initially marked before the process have increased (Figure 13).

Figure 13 – Points demarcated in the balloon and student explanations



Source: Research records (2020).

In letter c), students were asked to complete the comparative chart, a task that caused some difficulty. The question demanded more attention from the participants when comparing a situation in two different areas: the topological and the Euclidean. To guide this activity, each situation described on the balloon and on the sheet of paper was imagined. For example, draw an angle on the balloon and inflate it; then draw an angle on the paper, fold the sheet and ask: would the angle remain the same in both situations?

Some got lost in this activity, as it required a little more imagination and reasoning and, as it was the last period of class, there was a group that had to leave early, handing it in blank.

Activity 6 was answered by only 9 participants, and five of them did it correctly, four were wrong in at least one association. Among these four, one hit four associations; one hit three; one got two and one got one.

CONSIDERATIONS

In this qualitative research, the objective was to explore elementary topological properties, through activities involving concrete material resources. In this search, a way of inserting some preliminary notions about topological geometry in a 9th grade of elementary school was investigated. To achieve this goal, concrete materials were used, carefully elaborated by the researchers, so that properties of this geometry were discovered, although not formal from a mathematical perspective.

Through six activities ranging from the four-color problem to the Möebius Range, elementary properties of the topology were obtained. This is in line with what was said by Piaget and Inhelder (1993), that is, because it does not depend on measurements, topological geometry is more intuitive than the Euclidean one, with its well-defined metrics.

It is believed that the objective was achieved, since, at the end of the activities, the students realized, for example, the requirement of a minimum of four colors to paint a map so that neighboring states were not colored the same color. This problem has only recently been resolved with the advent of computational methods. This problem is relevant, for example, to obtain the relationship between vertices, faces and edges of a convex polyhedron. Thus, students with initial knowledge on the subject can be awakened and stimulated to the study of Euclidean spatial geometry in high school.

In turn, when exploring activities intuitively and concretely, obtaining the Möebius Strip, students may have notions more conducive to the study of finite, infinite, limited and unlimited sets, which may lead them to a better understanding of the study of progressions and limits in high school, as well as real numbers.

When investigating activities involving topological properties in an informal and recreational way, it is understood that there was an understanding of some of them, which allowed the investigated group to have the notion that not only Euclidean geometry needs to be studied and deepened. As there are numerous application possibilities, as exemplified in this article, it is argued that such activities could be included in Elementary School subjects.

The head teacher of the class and the author of the article hoped that the activities carried out would lead the students to be instigated to know other types of geometry and Mathematics. It was intended, with this, to go beyond those of the routine contents contained in the basic curriculum, without losing sight of its fulfillment. It was hoped, in this way, to arouse interest in geometry in its various nuances.

From these activities, it was noticed that the students were curious about the existence of geometric shapes not seen in Euclidean geometry, as well as properties that they didn't even imagine existed, such as the existence of the Möebius Tape, a unilateral surface. Furthermore, students discovered transformations of distinct objects while preserving topological properties. Figure 14 illustrates the concentration of students during the activities performed.

Figure 14 – Students performing topological activities

Source: Research data (2020).

The investigation showed that it is possible to use knowledge not used at the level in question, favoring learning, including concepts of Euclidean geometry from the constructions carried out, in addition to others of topological geometry. It is believed that the activities can also be relevant for other school levels, for example, in high school.

It is believed that the investigation accomplished what was desired and it is expected, subsequently, to apply something similar in high school classes, as it is expected that such innovations in the elementary school curriculum can awaken more individuals to Mathematics and, specifically, for geometry.

INVESTIGANDO PROPRIEDADES TOPOLÓGICAS COM ALUNOS DO ENSINO FUNDAMENTAL

RESUMO

Este artigo é resultado de uma pesquisa originada em um Grupo de Estudos e Pesquisas em Geometria, o qual tem se dedicado a explorar metodologias alternativas que possam contribuir com essa área, inclusive por meio da elaboração de recursos didáticos adequados, cuja aplicação dever acontecer *in locus*, a fim de validar e sugerir propostas. O que se apresenta neste trabalho teve por objetivo analisar como estudantes do Ensino Fundamental compreendem fatos de uma geometria não euclidiana - a geometria topológica - explorando recursos materiais didáticos concretos. Foram aplicadas seis atividades envolvendo recursos materiais como mapas para colorir com quantidade mínima de cores, massa de modelar, para concluir sobre a conservação de massa, cordões, para transformações contínuas de linhas fechadas, Banda de Möebius, para analisar a propriedade de continuidade, balões, para ilustrar deformações contínuas, dentre outros. Com isso, foi possível averiguar a compreensão dos alunos, a respeito do tema, nesse nível de escolaridade. Concluiu-se da investigação que é possível introduzir noções topológicas básicas que tendem a contribuir para a formação geométrica desses indivíduos, inclusive na passagem para a geometria do Ensino Médio, por exemplo, ao estudar a relação de Euler para poliedros.

PALAVRAS-CHAVE: Topologia. Ensino Fundamental. Banda de Möebius.

NOTES

1 http://revistavidroealuminio.com.br/centro-internacional-de-midia-phoenix-exemplo-de-construcao-sustentavel/. Access on: Jun. 10th,2020.

2 Daniel das Neves. Extracted from Galileo Magazine. The $1 million puzzle Issue 144-Jul/03, available on the internet: http://revistagalileu.globo.com/Galileu/0.6993,ECT560640-2680-2.00.html. Article by Carmem Kawano. Access on: Jun. 10th, 2020.

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