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Robust Digital Controllers via $\mathcal{H}_\infty$ Design for Lurie Type Systems

Rafael F. Pinheiro and Diego Colón

***Abstract*—The goal of this paper is to present a new methodology for the design of robust digital controllers for Lurie type systems via the mixed-sensitivity $\mathcal{H}_\infty$ technique with parametric uncertainties. For the purpose of comparing results, a well-known controller design approach via linear matrix inequalities with polytopic uncertainties is presented and used. An example is proposed, where numerical simulations are performed to ensure the effectiveness of the new controller, showing that the new methodology provides better results.**

***Index Terms*—Robust Digital Controller, Lurie Problem, Absolute Stability, Robust Control, $\mathcal{H}_\infty$ Control.**

I. Introduction

In 1944, due to a problem of automatic control of an aircraft, the Lurie problem was designed [1], which is also famous in the literature as an absolute stability problem. It started the groundwork for a significant field of the control engineering, which is Robust Control, running out on substantial contributions to mathematics and engineering.

Large number researchers have engaged in this problem for the years 50's and 60's, among whom can be cited in [2], with the well known Aizerman conjecture, Krasovskii (1953) [3], Popov (1961) [4], and Kalman (1963) [5]. On a path, investigation on the Lurie problem proceeds a larger jump in the 80's, when works initiate to spring up which connected the problem to other subjects and procedures such as chaos and chaos synchronization [6]; neural networks [7]; switched linear systems [8]; convex approach to the Lurie problem [9]; linear parameter varying (LPV) system [10]; uncertain systems [11]; and μ analysis [12], [13]. Additionally, its research remains contemporary in the aeronautic area, as can be viewed in [14].

To fix this problem, Nyquist criterion, Lyapunov functions and Linear Matrix Inequalities (LMI) are utilized. For example, the Circle criterion [15] uses the method of Nyquist; and the Popov criterion [4], typically less conservative than the Circle criterion, utilize Lyapunov functions in its demonstration. More recent works, like [16], [17], [18], [19], [10] also utilize Lyapunov functions and LMI.

On the other hand, the Lurie problem is a peculiar type of robust control problem, consequently, sophisticated robust control procedure like parametric uncertainty and structured singular value (SSV), due to the work's origin on [20], can be utilized for designing controllers in the Lurie structure. New works have emerged using Doyle's techniques [21]. Hence, this work starts the presentation of results in the area of digital control for the Lurie Problem using some techniques from Doyle's theory and also following Skogestad's ways [22], however, these do not present significant results for discrete systems. In addition, this paper is the result of suggestions for future research given by the articles [23], [24].

The main contribution of this work is the presentation of a new approach to the design of controllers for Lurie type systems in discrete time. It can be seen that in [25] a new design methodology is conducted via mixed-sensitivity $\mathcal{H}_\infty$ with parametric uncertainties, in the continuous time, with the suggestion to extend to discrete time. Thus, this work presents the discrete time extension of design of controllers via mixed-sensitivity $\mathcal{H}_\infty$ for Lurie type systems with parametric uncertainties. Regarding the new applications of this theory, we visualize the possibility of great contribution in the areas of health and renewable energy.

In order to perform a comparison of methodologies, an approach that uses LMI and polytopic uncertainties was presented. There are numerous works that use this methodology for Lurie type systems ([26], [27], [28]). Here the technique is presented with the theoretic basis from the: [29], [30], [31], [32] and [33]. A simple example is given for the purpose of comparison and didactic understanding of the theory presented in this work.

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The remaining of this work is organized as follow: In sections II and III, we treat in continuous-time to understand the problem and define the necessary tools to the approaches presented in this paper. The main results are given in Section IV, where we introduce discretization and design methods. In Section V, we deal with a numerical example and finally, in Section VI, we have the conclusion that summarizes the main discoveries and proposes future research.

II. PROBLEM STATEMENT

The Lurie problem in SISO case, or absolute stability problem, consists in finding necessary and sufficient conditions to the global asymptotic stability of the system of Figure 1. In this system, we have a linear dynamics represented by the system L, and the function $f(\sigma)$ represents a nonlinearity.

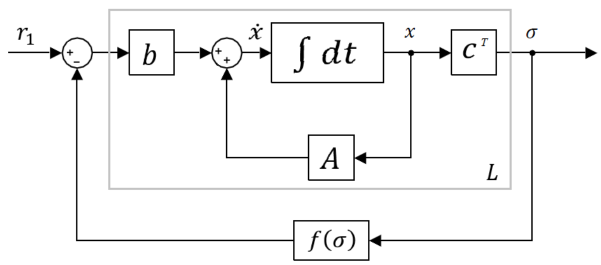

Fig. 1. Block diagram of the Lurie type system.

Considering $r_1 = 0$, in (1) we have a system of differential equations, which was known in the literature as Lurie type system:

$$\begin{cases} \dot{x} = Ax - bf(\sigma) \\ \sigma = c^T x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, and $b, c \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ are fixed matrices. Generally, the nonlinearity $f(\sigma)$ is a continuous function restricted to the first and third quadrants of the plane (see Figure 2). It belongs to one of the following families:

$$F_{(0,k]} := \{f \mid f(0) = 0, 0 < \sigma f(\sigma) \leq k\sigma^2, \sigma \neq 0\},$$

$$F_{(0,k)} := \{f \mid f(0) = 0, 0 < \sigma f(\sigma) < k\sigma^2, \sigma \neq 0\},$$

$$F_{[k_1,k_2]} := \{f \mid f(0) = 0, k_1\sigma^2 \leq \sigma f(\sigma) \leq k_2\sigma^2, \sigma \neq 0\},$$

$$F_\infty := \{f \mid f(0) = 0, \sigma f(\sigma) > 0, \sigma \neq 0\}.$$

Note that, the functions f may be not precisely known, and pertaining to an uncertain family, which justifies why the Lurie problem can be considered one of the founding of robust control.

III. THEORETICAL BACKGROUND

This section sets forward the essential tools to build the main contribution of the paper.

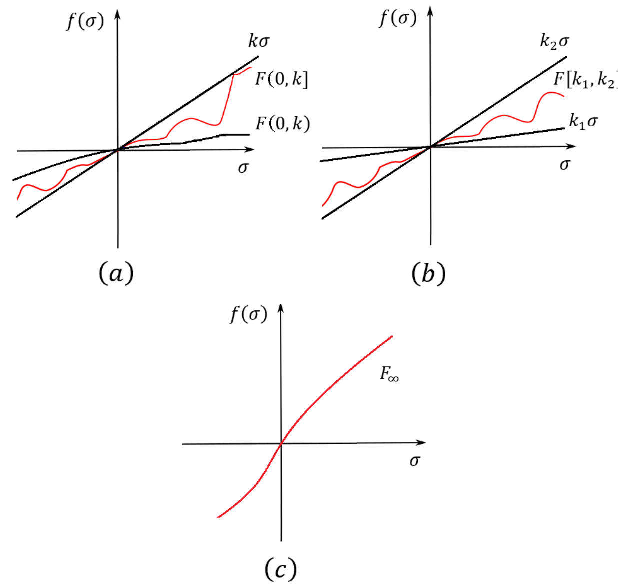

Fig. 2. Types of functions f: a) $F_{(0,k]}$, $F_{(0,k)}$; b) $F_{[k_1,k_2]}$; c) F_∞.

A. $\mathcal{H}_\infty$ Mixed-Sensitivity Method

This procedure is well known in the literature and a large number of works has tackled similar problems like in [20], [22] and [34]. It utilize sensitivity function $S = (I + G_oK)^{-1}$ and $T = (I + G_oK)^{-1}\ G_oK$ complementary sensitivity for the design of controllers.

We intend to design a controller via mixed-sensitivity $\mathcal{H}_\infty$ (S/KS/T), where the objective is to minimize the $\mathcal{H}_\infty$ norm of (2):

$$\|N\|_\infty = max_\omega \gamma(N(j\omega)) < 1; \quad N = \begin{bmatrix} W_pS \\ W_uKS \\ W_IT \end{bmatrix}, \tag{2}$$

where γ, in (2), indicate $\mathcal{H}_\infty$ upper bound, which may be interpreted as the measure of the matrix N at any particular frequency. W_p, W_u and W_I are weight functions. The weight W_I is utilized for modeling the uncertainty and is specific by the uncertainty model of the family of plants. The weights W_u and W_p are selected by the designer to reflect the expected specifications for the closed loop system. W_u constitute a weight for the control effort. The weight W_p was selected as introduced in [22], and depicted as follows:

$$W_p = \frac{s/M + w_b}{s + w_bA}, \tag{3}$$

where: ω_b is the minimal bandwidth frequency (specified as the frequency where $S(j\omega)$ crosses from below); A is parameter for $S(s)$ to be little in low frequency, this causes a little stationary error, doing $A \leq 1$; and M is maximal peak magnitude of S, $\|S(j\omega)\|_\infty$. Typically $M = 2$.

In respect of uncertainties modelling, [22] gives multiple alternative for the designer to select the type of uncertainty that is most appropriate for the design. In Figure 3, the uncertainty

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model known as inverse multiplicative input is introduced and defined as:

$$G_p(s) = \frac{G_o(s)}{1 + W_I(s)\Delta_I(s)}, \quad \|\Delta_I\|_\infty \leq 1. \tag{4}$$

The transfer function $G_o(s)$ is the nominal model of the plant and $G_p(s)$ constitute the disturbed plant, which means the uncertainties are considered in $G_p(s)$. The variable Δ, in (4), indicates the normalized perturbation with the $\mathcal{H}_\infty$ norm less or equal than 1.

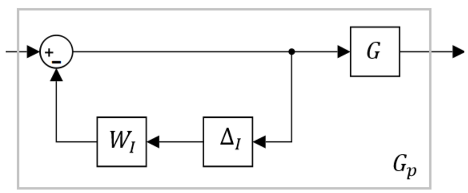

Fig. 3. Inverse multiplicative input uncertainty.

In Figure 4, the extended plant is presented, included the weights W_p, W_I e W_u. In this method, the weights are the fine tuning tools utilized to choose the best adjustment that suffices the request for a specific situation.

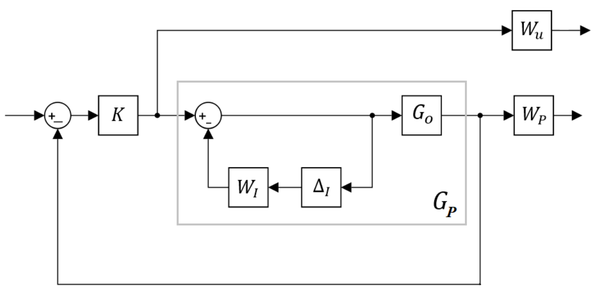

Fig. 4. Control diagram with weights.

The sub-optimal controller $\mathcal{H}_\infty$ is achieved by solving the optimization problem:

$$min_K \|N(K)\|_\infty, \tag{5}$$

where K constitute a stabilizing controller.

Remark that, a similar method may be applied in the case (S/KS), meaning that the following constraints must be examined:

$$\|N\|_\infty = max_\omega \gamma(N(j\omega)) < 1; \quad N = \begin{bmatrix} W_p S \\ W_u KS \end{bmatrix}. \tag{6}$$

B. $\|H_\infty\|$ State Feedback Controller Design via LMI

The control law can be defined as:

$$u(m) = Kx(m). \tag{7}$$

In [30], the LMI constraint to design a state feedback controller (7) is presented. The controller can be designed using the following theorem:

Theorem 1: There exists a controller in the form (7) such that the inequality $\|H\|_\infty^2 < \gamma$ holds if, and only if, the LMI:

$$\begin{bmatrix} P & AX + BL & J & 0 \\ \bullet & X + X^T - P & 0 & X^T C_z^T + L^T D_z^T \\ \bullet & \bullet & I & D_z^T \\ \bullet & \bullet & \bullet & \gamma I \end{bmatrix} > 0, \tag{8}$$

holds, where the matrices X and L and the symmetric matrix P are the variables. The controller is obtained by $K = LX^{-1}$.

The proof for this theorem can be found in [30].

C. Polytopic Uncertainty

A system subjected to uncertainties can be represented as a polytopic system as follows:

$$\mathcal{G}_a : \begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(k) \\ y(t) = C(t)x(t), \end{cases} \tag{9}$$

and if these parameters with uncertainty vary within a known range, it is possible to describe the system matrices as vertex belonging to a polytope. Each vertex represents a linear system with a specific combination of the uncertain parameters.

D. Modeling Nonlinearity by Means of Uncertainties

1) Parametric Uncertainty: Over the years, many necessary and sufficient conditions for the absolute stability of Lurie type systems have been derived, as in [6], [7]. Now assume that we wish to enhance the stability and performance on a Lurie type system. The following is a methodology that has been obtained from [25].

To transform the block diagram of Figure 1 into a similar form of Figure 4, we replace f by a parametric uncertainty. This new system can be interpreted as an equivalent family of plants G_p.

Let f be as exhibited in Figure 5. The idea is to substitute f by a set of linear functions $k\sigma$, where k is the uncertainty. In practice, the block $f(\sigma)$ can be replaced by the uncertain block gain α. From the diagram of Figure 1, we acquire the perturbed system as follows:

$$\dot{x} = Ax + b(r_1 - \alpha c^T x) \quad \rightarrow \quad \dot{x} = Ax + br_1 - b\alpha c^T x$$

$$\rightarrow \quad \dot{x} = (A - b\alpha c^T)x + br_1.$$

Making $A_\alpha = (A - b\alpha c^T)$, where $\alpha = \frac{k}{2} + \frac{k}{2}\delta$, $\mid \delta \mid \leq 1$, we have:

$$\begin{cases} \dot{x} = A_\alpha x + br_1 \\ \sigma_\alpha = c^T x. \end{cases} \tag{10}$$

Later, we will use the system (10) to obtain the transfer function in discrete time.

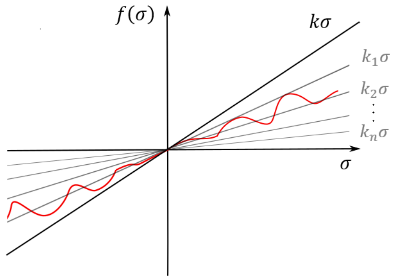

Fig. 5. Mapping $0 < f(\sigma) \leq k\sigma$.

2) Polytopic Uncertainties: Consider that the nonlinearity is bounded in a section in the first and third quadrants bounded by a frontier k, as introduced in the Figure 5. The change to a polytopic uncertainty is designated by the following equations:

$$A_l = A + B\rho C, \tag{11}$$
$$\rho \in [0, k]. \tag{12}$$

where A_l constitute the polytope vertex. Broadly the concept above for a system where there is more than one nonlinearity, the equation to obtain the polytope vertex is given by:

$$A_l = A^{n\times n} + B^{n\times q}\begin{bmatrix}\rho_{11} & \cdots & \rho_{1q}\\ \vdots & \cdots & \vdots \\ \rho_{q1} & \cdots & \rho_{qq}\end{bmatrix}^{q\times q} C^{q\times n}. \tag{13}$$

With the generalization introduced in (13) it is feasible to establish a vertex for each combination of nonlinearity, and also it is possible to think about some coupling between the nonlinearity. After obtaining the vertex, the polytope is obtained using the following equations:

$$\mathcal{A}(\beta) = \left\{\sum_{l=1}^{V}\mu_l A_l, \quad \mu_l \geq 0, \sum_{l=1}^{V}\mu_l = 1\right\} \tag{14}$$

It is simple to employ the polytope in the LMI constraint exhibit in the Theorem 1. The method is simply adding the index l in the LMI constraint presented in the Theorem 1:

$$\begin{bmatrix} P & A_l X + B_l L & J & 0 \\ \bullet & X + X^T - P & 0 & X^T C_z^T + L^T D_z^T \\ \bullet & \bullet & I & D_z^T \\ \bullet & \bullet & \bullet & \gamma I \end{bmatrix} > 0, \tag{15}$$

where the insertion of the index l means that there are now l LMI constrains, instead of only one constraint, which means an increase of conservatism in the optimization problem. For that reason, the Theorem 1 loses the necessary condition, and now provides only sub-optimal condition. And, as the number of vertices increases, the optimization problem feasibility decreases.

IV. Discretization Procedures and Method of Obtaining the Controllers

For the discretization process and obtainment of the controllers, a class of Lurie type systems (1) with one nonlinearity and the matrix A Hurwitz and for $\sigma \neq 0$, $0 < f(\sigma) \leq k\sigma$, and for $\sigma = 0$, we have $f(0) = 0$.

A. Discretization with Parametric Uncertainties

Supposing the existence of parametric uncertainties in the Lurie system in the discrete-time domain, the Zero-Order-Holder (ZOH) is not appropriate for the task, due to the fact that the information about the nonlinearity may be lost.

Consequently, for the Lurie problem, a discretization using the finite difference methods fits completely. This is also known as the Euler method by difference in forward rectangular. So, a differential equation is transformed into a finite difference equation:

$$\dot{x}(n) \cong \frac{x[n+1] - x[n]}{T_s}. \tag{16}$$

So, (10) in the discrete form becomes:

$$\left\{ \; \tfrac{x[n+1]-x[n]}{T_s} = A_\alpha x[n] + br_1[n], \right. \tag{17}$$

knowing that $x[n+1] = zX(z)$, $x[n] = X(z)$ and $r_1[n] = R_1(z)$, we get the z-transform of the preceding system, its transfer function, and finally the inverse multiplicative input uncertainty:

$$G_p(z) = \frac{X(z)}{R_1(z)} = \frac{G(z)}{1 + W_I(z)\Delta_I(z)}, \quad ||\Delta_I||_\infty \leq 1. \tag{18}$$

Thereby, the Lurie type system in continuous-time of Figure 1 is modify into the discrete-time system of Figure 6.

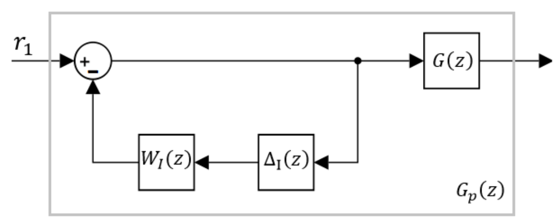

Fig. 6. Lurie type system in discrete form.

The $\mathcal{H}_\infty$ sub-optimal discrete-time control problem is to discover an internally stabilizing controller $K(z)$ such that, for a pre-specified positive value γ:

$$||F_L(P, K)||_\infty < \gamma. \tag{19}$$

N of equation (6) is connected to P and K by a lower linear fractional transformation (F_L), as follows:

$$F_L(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} = N, \tag{20}$$

in such a way that link equations (19) and (6). The design of the generalized plant P is clarified in [22]. The controller is obtained using the Matlab functions $hinfsyn$ or $mixsyn$. These functions obtain the controller solving two Riccati equations in order to satisfy conditions of asymptotic stability for the control problem (see [35]).

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B. Discretization with Polytopic Uncertainties

The discretization procedure is a routine method and there are some techniques to perform it, like the Zero-Order-Holder. Yet, the commonly used discretization process imposes some challenges for polytopic systems. That happens as an effect of the polytope derived by discretizing all the continuous-time matrices belonging to a given polytopic system, which can result in the loss of convexity, making impossible to use any convex optimization process to design the controllers.

Thus, common method like the Zero-Order-Holder becomes inadequate for the task. Nevertheless, there are some proposals in the literature to address this challenge, like [29] and [31]. Those method are numerical solutions or Taylor series expansions of the exponential matrices connected to the discretization with a finite number of terms. In the current paper, we use the solution provided by [33], which is a Taylor series expansion with variable number of terms l. This feature is convenient, because it is necessary to bound system norm and the residue norm to assure the performance. For that, we employ the Small Gain Theory. The proof can be found in [32]. Taking into account the continuous-time polytopic system:

$$\dot{x}(t) = A_{cont}(\beta)x(t) + B_{cont}(\beta)u(t), \tag{21}$$

the equivalent discrete-time polytopic system is:

$$x(n+1) = A_{disc}(\beta)x(n) + B_{disc}(\beta)u(n), \tag{22}$$

where it is possible to write the discrete matrices as:

$$\begin{aligned} A(\beta) &= A_l(\beta) + \Delta A_l(\beta), &(23)\\ B(\beta) &= B_l(\beta) + \Delta B_l(\beta), &(24)\end{aligned}$$

where $A_l(\beta)$, $B_l(\beta)$ are the Taylor series expansion and $\Delta A_l(\beta)$, $\Delta B_l(\beta)$ Taylor series residue. The matrices $A_l(\beta), B_l(\beta), \Delta A_l(\beta)$ and $\Delta A_l(\beta)$ can be calculated as:

$$\begin{aligned} A_l(\beta) &= \sum_{j=0}^{l} \frac{A_{cont}(\beta)^j}{j!}T^j, &(25)\\ B_l(\beta) &= \sum_{j=1}^{l} \frac{A_{cont}(\beta)^{j-1}}{j!}T^j B_{cont}(\beta), &(26)\\ \Delta A_l(\beta) &= e^{A_{cont}(\beta)T} - A_l(\beta), &(27)\\ \Delta B_l(\beta) &= \left(\int_0^T e^{A_{cont}(\beta)s}ds\right) B_{cont}(\beta) - B_l(\beta) &(28)\end{aligned}$$

where $l \in \mathbb{N}$ is the number of terms in the Taylor expansion and $T > 0$[s] is the sampling time. After expanding in Taylor series the polynomial matrices must be homogenized, for details see [33].

V. Numerical Example

For the purpose of illustrating the results, comparisons, and verification of the effectiveness of the method, we present this example, which aims to design robust digital controllers according to Figure 7, for the Lurie system (29) with $T_s = 0.1$ and $k = 2$ (in this case $f \ inF_{(0,2]}$).

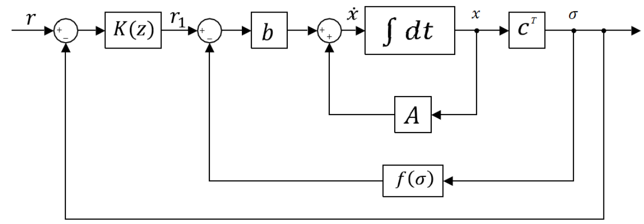

Fig. 7. Block diagram of the numerical example.

$$\begin{cases} \dot{x}_1 = -x_1 + f(\sigma) \\ \dot{x}_2 = -x_2 \\ \sigma = x_1. \end{cases} \tag{29}$$

The matrices for (29) are:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad and$$

$$c^T = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The example is divided into two parts. Part A contains the design of controllers using the new methodology, that is, mixed-sensitivity $\mathcal{H}_\infty$, which was subdivided into design (S/KS) and $(S/KS/T)$ with Parametric Uncertainties. Part B contains the design via LMI and polytopic uncertainties, which is used to compare with the new methodology. In all designs, time simulations were performed.

A. Design via Mixed-Sensitivity $\mathcal{H}_\infty$ (S/KS) and $(S/KS/T)$ with Parametric Uncertainties

First, we replace $\alpha = \frac{k}{2} + \frac{k}{2}\delta$, $\mid \delta \mid \leq 1$. So, using (17) and (18) we get the weight for uncertainty and the nominal plant $G(z)$, with this, we have: $G(z) = \frac{Ts}{z-1+Ts+\frac{kTs}{2}}$ and $W_I(z) = \frac{\frac{kTs}{2}}{z-1+Ts+\frac{kTs}{2}}$.

1) Controller design via (S/KS) approach: In order to obtain W_p, we use (3) and to its parameters we choose $M = 2$, $A = 0.005$ and $w_b = 10$ as suggested in [22]. Using the same method as presented in section III-A $W_p(z) = \frac{0.5z+0.4975}{z-0.995}$ is obtained. We discretize W_p using the function $c2d$ of the Matlab. For the penalization of control, we choose: $W_u = 0.0561$. This value was achieved in a few tries, until we find the best fit. After achieving the plant P and using the Matlab function $hinfsyn$, the following controller is obtained:

$$K(z) = \frac{9.932z^4 - 15.84z^3 - 3.605z^2 + 15.84z - 6.325}{z^4 - 1.792z^3 - 0.2032z^2 + 1.786z - 0.7907}, \tag{30}$$

with achieved gamma value: $\gamma = 1.0046$.

2) Time Simulation (S/KS): The first simulation is the step response of some plants in the family of parameters $0 < \alpha \leq 2$ of G_p randomly chosen. In Figure 8, it is feasible to note that the controller behaves similarly for all the plants examined. Following, we conducted simulations via Simulink with nonlinearity equal to a hyperbolic tangent, as exhibited in Figure 9. Herewith, we obtain responses to sine and step wave of amplitude equal to 5, as shown in Figure 10 and Figure 11.

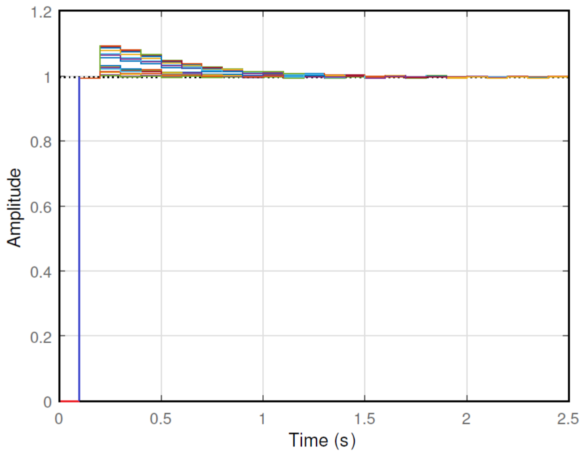

Fig. 8. Step response of the family of plants with S/KS controller for $0 < \alpha \leq 2$.

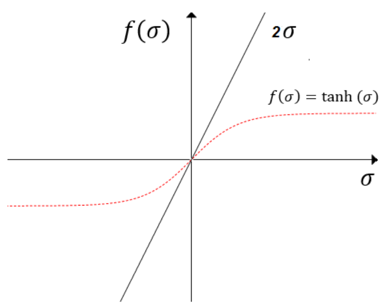

Fig. 9. The hyperbolic tangent.

Note that the controller behaves as expected, following the reference signal. Even with the presence of hyperbolic tangent as nonlinearity, the controller still tracks follows the reference in both simulations.

3) Controller Obtained $(S/KS/T)$: In order to obtain W_p we use (3) and for its parameters we did a little different than proposed in [22]: we have selected $M = 5$, $A = 0.005$ and $w_b = 1\ rad/s$, and with a fine adjustment of these parameters we get a good value for the norm. We discretize W_p using the function $c2d$ of Matlab, obtaining: $W_p(z) = \frac{0.2z-0.1}{z-0.9995}$. For the control penalization, we opt for $W_u = 0.01$; this value was also obtained in a few tries. According to the Matlab functions $mixsyn$ we obtain the controller:

$$K(z) = \frac{7.497z^3 - 4.397z^2 - 7.358z + 4.862}{z^3 - 1.024z^2 - 0.5958z + 0.6201}, \tag{31}$$

with gamma value (norm): $\gamma = 1.0064$.

4) Time Simulation $(S/KS/T)$: In a similar manner to the (S/KS) design, we test the robustness of the controller. We have in the Figure 12 the response to the unitary step, for not only the nominal plant G, but for other plants randomly chosen according to the uncertainty parameter $0 < \alpha \leq 2$ of G_p.

Correspondingly, to the previous simulation, the controller works as planned for all the tested plants. In simulations via Simulink with nonlinearity equal to the hyperbolic tangent, we have in Figure 13 and Figure 14 the responses to sine and step inputs with amplitude 5.

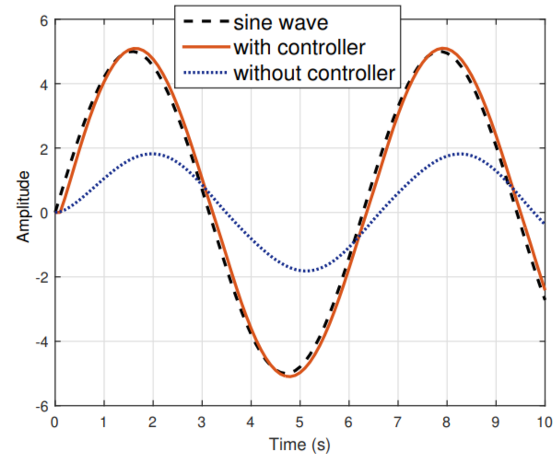

Fig. 10. Response to sine wave with digital controller by mixed-sensitivity (S/KS) with continuous plants.

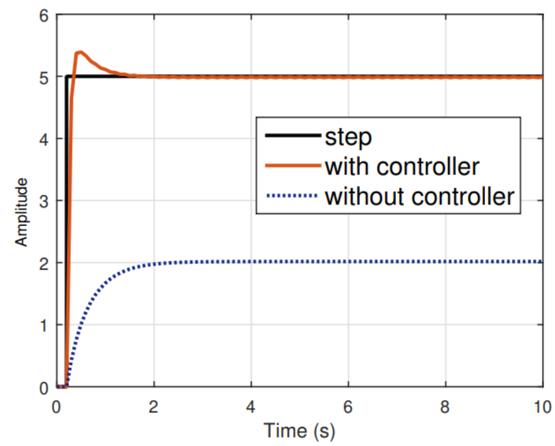

Fig. 11. Response to step wave with digital controller by mixed-sensitivity (S/KS) with continuous plants.

B. Design via LMI and Polytopic Uncertainties

1) Controllers obtained: The Theorem 1 is used to obtain the controller and also the gamma value.

	$\|H\|_\infty$ ***Controller via LMI***	gamma value
$l = 2$	[-16.4241]	1.4805
$l = 3$	[-24.6527]	1.0059
$l = 4$	[-29.4739]	1.2421

TABLE I
CONTROLLER OBTAINED USING THEOREM 1 FOR $l \in [2, 3, 4]$.

Table I shows the number l (LMI restrictions) and the obtained norm. Note that for a $l = 3$, although it increases the conservatism slightly, we have a smaller value of the norm.

2) Temporal Simulation: We use to simulate nonlinearity the function $f(\sigma) = tanh(\sigma)$, as shown in Figure 9. Differently from the two previous cases, the controller using the LMI approach as shown in Figure 15 presented a higher error, on the other hand, there is no presence of overshooting.

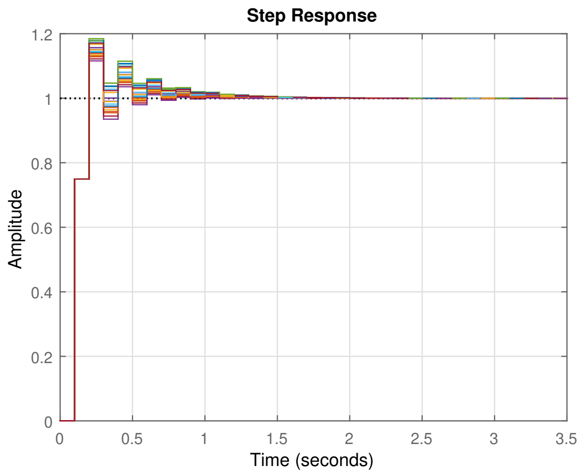

Fig. 12. Step response of the family of plants with $S/KS/T$ controller for $0 < \alpha \leq 2$.

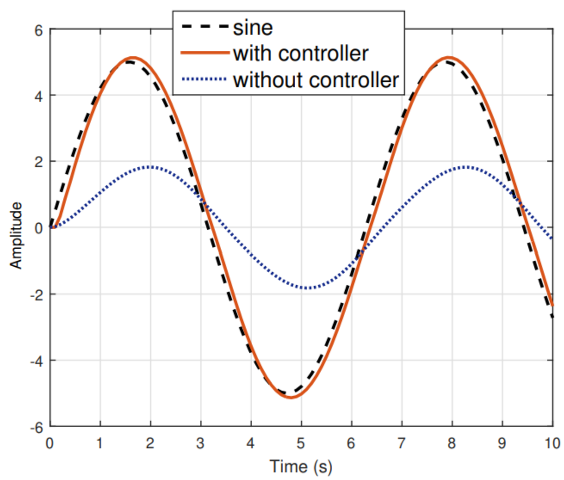

Fig. 13. Response to sine wave with digital controller by mixed-sensitivity $(S/KS/T)$ with continuous plants.

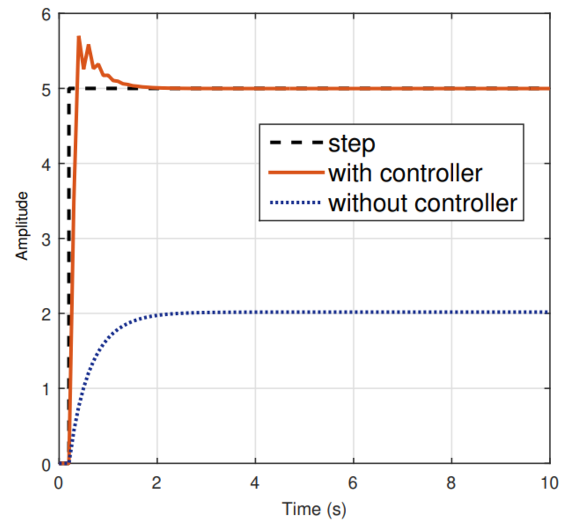

Fig. 14. Response to pulse wave with digital controller by mixed-sensitivity $(S/KS/T)$ with continuous plants.

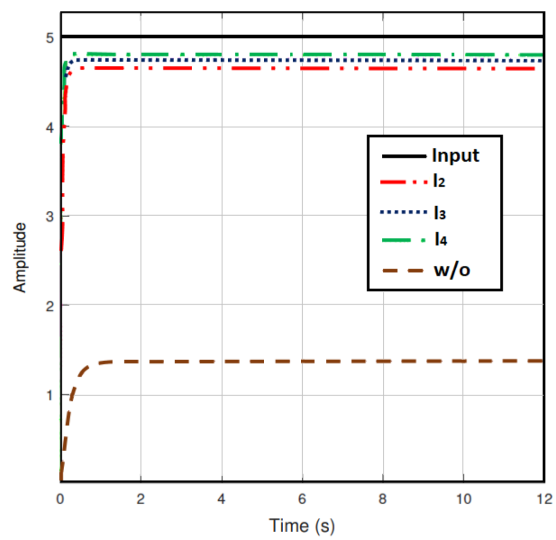

Fig. 15. Graphics obtained using all the controllers designed via LMI and polytopic uncertainties, and without controller (w/o).

VI. Conclusion

In this work, we developed a new procedure for obtaining robust digital controller for a class of SISO Lurie type systems via mixed-sensitivity $\mathcal{H}_\infty$. For comparison, we designed a controller via LMI with polytopic uncertainties, which is a well-known approach in the literature. This controller, for the proposed example, presented a non-removable error, however, the problem could be easily solved with an integrator. The controller via mixed-sensitivity $\mathcal{H}_\infty$ (S/KS) presented a smooth oscillation in steady-state, which we could not remove. On the other hand, the controller via mixed sensitivity $\mathcal{H}_\infty$ $(S/KS/T)$ showed no significant oscillation or error and, in our analysis, performed best for the proposed example.

As a limitation of the method presented, we can consider that there is not yet a general procedure for any type of plant. We intend, in future work, to solve this question similarly as done for the continuous case in [24], on the way to prove stability and performance robustness in any frequency band. Also, as future work, we are planning to develop a procedure to obtain a digital controller for multiple inputs, similar to what was done in [23] for the continuous case.

In terms of applications, the following are suggestions for future works in relevant areas:

- Application in the area of renewable energies. The results constituted in this work may be helpful in the designing robust controllers for power take-off (PTO) of wave energy converters (WEC). It is noted in papers such as [36], [37], [38], [39], [40] the demand of robust controllers, because in WEC-PTO systems, there are many nonlinearities (which can be modelled as uncertainties, see [23], [24], [25]) in actuators, valves, accumulators, temperature variations, and sea conditions.
- To apply the method to a discrete Hopfield neural network (as made by [41]), which is used to simulate associative

memory. This memory can be considered analogue to the human memory, and so it can be used for modeling a neuropathology, for example the Alzheimer's disease. In this context, the controller could be used to assuage the effects of the disease.
- Another application in the healthcare area, might be for controlling cardiac devices [42], [43], [44], since there are many variables with uncertainties in this area.

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Controladores Digitais Robustos via projeto $\mathcal{H}_\infty$ para Sistemas do Tipo Lurie

***Resumo:* —O objetivo deste trabalho é apresentar uma nova metodologia para o projeto de controladores digitais robustos para sistemas do tipo Lurie via técnica da sensibilidade-mista $\mathcal{H}_\infty$ com incertezas paramétricas. Para fins de comparação de resultados, uma conhecida abordagem de projeto de controladores via desigualdades matriciais lineares com incertezas politópicas é apresentada e utilizada. Um exemplo é dado onde são realizadas simulações numéricas que garantem a eficácia do novo controlador, mostrando que a nova metodologia proporciona melhores resultados.**

***Palavras-chave:* —Controlador Digital Robusto, Problema de Lurie, Estabilidade Absoluta, Controle Robusto, Controle $\mathcal{H}_\infty$.**

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